86. *Information and Statistics.* II

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This is a continuation of Kawada [0]. We use the same notations.

II. **L-sets and informations.** **1.** Let $p=(p_1, \cdots, p_m)$ and $q=(q_1, \cdots, q_m)$ be probability distributions. We call the set

$$L(p, q)=\left\{(x, y)\middle| x=\sum_{k=1}^{m} \alpha_k p_k,\ y=\sum_{k=1}^{m} \alpha_k q_k,\ 0 \leqq \alpha_k \leqq 1,\ k=1, \cdots, m\right\} \tag{9}$$

the *Liapunov-set* (simply *L-set*) of the pair (p, q). See Kudō [6], [7].

$L(p, q)$ has the following properties:

(i) $L(p, q)=\Delta$ (the diagonal segment joining (0, 0) and (1, 1)) if and only if $p=q$.

(ii) $L(p, q)$ contains the points (0, 0) and (1, 1).

(iii) $L(p, q)$ is contained in the square $[0, 1]\times[0, 1]$.

(iv) $L(p, q)$ is a symmetric convex set with the center (1/2, 1/2).

(v) Let the indices of (p_k, q_k) be so substituted that

$$0 \leqq (q_1/p_1) \leqq (q_2/p_2) \leqq \cdots \leqq (q_m/p_m) \leqq \infty$$

holds. Then

$$L(p, q)=\{(x, y) \mid \varphi(x) \leqq y \leqq \psi(x),\ 0 \leqq x \leqq 1\}$$

where $\varphi(x)$ is a polygon with $m+1$ vertices

$$(0, 0),\ (p_1, q_1),\ (p_1+p_2, q_1+q_2), \cdots, (p_1+\cdots+p_{m-1}, q_1+\cdots+q_{m-1}),\ (1, 1)$$

and $\psi(x)$ is a polygon with $m+1$ vertices

$$(0, 0),\ (p_m, q_m),\ (p_m+p_{m-1}, q_m+q_{m-1}), \cdots,$$
$$(p_m+p_{m-1}+\cdots+p_2,\ q_m+q_{m-1}+\cdots+q_2),\ (1, 1).$$

Theorem 6. *A function $I(p, q)$ for any pair of finite probability distributions (p, q) is an information if and only if*

(i) $L(p, q)=\Delta \Rightarrow I(p, q)=0$,

(ii) $L(p, q)=L(p', q') \Rightarrow I(p, q)=I(p', q')$,

(iii) $L(p, q) \supsetneqq L(p', q') \Rightarrow I(p, q)>I(p', q')$.

Namely, an information I is characterized by the property that I is a monotone functional of the family of all L-sets with $I=0$ for $L=\Delta$.

2. (i) We can characterize a fundamental information I geometrically as

$$I_K(p, q)=\int_C K(d\varphi/dx)dx \tag{10}$$

where $K(x)$ is a non-negative differentiable function with $K(1)=K'(1)=0$, $K''(x)>0$, $\varphi(x)$ is the polygon defined as above and the integral is the curvilinear integral along the polygon $C: y=\varphi(x)$.

In particular, if we put