69. On the Existence and Asymptotic Behavior of Solutions of Nonlinear Heat Flow with Memory

By Nobuyuki KATO
Department of Mathematics, School of Education, Waseda University

(Communicated by Kôsaku YOSIDA, M. J. A., Sept. 14, 1987)

1. Introduction and result. We shall consider the problem of nonlinear heat flow in materials with memory:

$$(\mathrm{M})\begin{cases}\dfrac{\partial}{\partial t}\left[u(t,x)+\displaystyle\int_{-\infty}^{t}k(t-s)u(s,x)ds\right]=\sigma(u_x(t,x))_x+h(t,x),\\ \qquad\qquad t\in \boldsymbol{R}^+,\ x\in(0,1),\\ u_x(t,0)\in\beta_0(u(t,0)),\quad -u_x(t,1)\in\beta_1(u(t,1)),\quad t\in\boldsymbol{R},\\ u(t,x)=u_0(x),\quad t\in(-\infty,0],\quad x\in(0,1).\end{cases}$$

Throughout, k, σ and β_i $(i=0,1)$ are always assumed that

(k) $k\in L^1(0,\infty)$, nonnegative, nonincreasing and bounded.

(σ) $\sigma\in C^1(\boldsymbol{R})$, $\sigma(0)=0$, $\sigma(\boldsymbol{R})=\boldsymbol{R}$, and σ is strictly increasing.

(β) $\beta_i(i=0,1)$ are maximal monotone graphs in $\boldsymbol{R}\times\boldsymbol{R}$ satisfying $0\in\beta_i(0)$.

Our purpose is to obtain the following

Theorem 1.1. *Let $h\in L^1(0,\infty;L^p(0,1))$ and $u_0\in L^p(0,1)$, $1<p<\infty$. Assume that the one of the following conditions is satisfied:*

(A) *$\beta_i\equiv 0$ for $i=0$ and 1.*

(B) *σ satisfies $\sigma'>0$ and*

$$(1.1)\qquad \int_0^\infty r\cdot\min\{\sigma'(s):|s|\le r\}dr=\infty,\quad \textit{in addition to } (\sigma),$$

and β_i satisfies

$$(1.2)\qquad \sup\{|y|:y\in R(\beta_i)\}<\infty\quad \textit{for } i=0 \textit{ or } 1\ (R \textit{ means a range}).$$

(C) *σ satisfies*

$$(1.3)\qquad \exists\delta>0:\sigma'\ge\delta,\quad \textit{in addition to } (\sigma).$$

Then the unique "generalized solution" $u(t,x)$ of (M) *(defined below) exists and it converges strongly in $L^p(0,1)$ to some constant ζ_∞ satisfying $0\in\beta_i(\zeta_\infty)$ $(i=0,1)$ as $t\to\infty$.*

Remarks. 1) The condition (1.1) was introduced by [11] and it states roughly that the gradient of σ is allowed to lie to some extent. Note that (1.3) implies (1.1).

2) In the case of (A), it is easy to see that

$$\zeta_\infty=\int_0^1 u_0(x)dx+\left(1+\int_0^\infty k(s)ds\right)^{-1}\int_0^\infty\int_0^1 h(t,x)dxdt\qquad (\text{cf. [1]}).$$

3) In the case of Dirichlet boundary condition, if (1.3) is assumed, we can obtain the estimate of decay corresponding to an exponential decay ([3], [7]):

$$(1.4)\qquad \|u(t)\|_p\le\left(\int_t^\infty r(\tau)d\tau\right)\|u_0\|_p+\omega^{-1}\int_0^t r(t-\tau)[u(\tau),h(\tau)]_+d\tau,$$