68. On Meromorphic and Univalent Functions

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1. Introduction. In the previous paper [1] We derived the area-principle on meromorphic and univalent functions in an annulus and then showed some properties of such functions. In the present paper we shall give the above-mentioned area-principle in the precise form where the omitted area (hereafter defined in Theorem 1) is considered and then improve some results in [1]. Moreover we shall deal with the case of meromorphic and univalent functions in the unit circle $|z|<1$, by means of the results in the case of an annulus.

2. We consider the following annulus

$$D : r<|z|<1 \quad (r>0).$$

Let $w=\varphi_\theta(z,\zeta)$ be regular in D, except for a simple pole of residue 1 at $\zeta\in D$ and univalently map D onto the whole w-plane with two parallel rectilinear slits of the inclination θ. Then $\varphi_\theta(z,\zeta)$ is given as follows ([6], p. 375)

$$\varphi_\theta(z,\zeta)=N(z,\zeta)+e^{i2\theta}M(z,\zeta)$$

where

$$N(z,\zeta)=\frac{1}{z-\zeta}+\frac{1}{\zeta}\sum_{n=1}^{\infty}\frac{r^{2n}((z/\zeta)^{-n}-(z/\zeta)^n)}{1-r^{2n}}=\frac{1}{2}(\varphi_0+\varphi_{\pi/2}).$$

$$M(z,\zeta)=\frac{1}{\bar\zeta}\sum_{\substack{n=-\infty\\(n\neq 0)}}^{\infty}\frac{(z\bar\zeta)^n}{1-r^{2n}}=\frac{1}{2}(\varphi_0-\varphi_{\pi/2}).$$

We shall give the improved area-principle in the case of an annulus.

Theorem 1. *Let $f(z)$ be regular, except for a simple pole of residue* 1 *at $\zeta\in D$ and univalent in the annulus D. Let δ denote the area of the complementary set of the image domain under $w=f(z)$. (We call δ the omitted area (cf. [4], [7]).) Moreover let $f(z)-N(z,\zeta)=\sum_{n=-\infty}^{\infty}a_nz^n$ in the annulus D. Then we have the following equality.*

$$\sum_{n=-\infty}^{\infty}n(1-r^{2n})|a_n|^2=\pi K(\zeta,\zeta)-\frac{\delta}{\pi},$$

where

$$K(z,\zeta)=\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{n(z\bar\zeta)^{n-1}}{1-r^{2n}}.$$

denotes the Bergman's kernel function of D.

Proof. We may consider the results in [1] or [5].

Corollary 1. *Let $w=f(z)$ satisfy the same conditions in Theorem* 1 *and δ denote the omitted area of $w=f(z)$. Then we have the following inequality.*