66. Harmonic Analysis on Negatively Curved Manifolds. I

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Let M be a complete, simply connected, n-dimensional Riemannian manifold of the sectional curvature K_M satisfying $-b^2 \leq K_M \leq -a^2$ for some constants $a, b>0$. The basic aim of this series of papers is to generalize the harmonic analysis on the open unit disc to the manifold M. In the first paper we treat Hardy spaces H^p, the space BMO and their probabilistic counterparts defined on the sphere at infinity $S(\infty)$ of M. Our research will deeply depends on recent remarkable works of M. T. Anderson, R. Schoen and D. Sullivan ([1], [2], [6]).

1. H^p and BMO. Throughout the paper we fix a point o in M. Let $\bar{M}=M \cup S(\infty)$. For $p \in M$ and $x \in \bar{M}$, we denote by $\gamma_{p,x}$ the uniquely determined unit speed geodesic ray with $\gamma_{p,x}(0)=p$ and $\gamma_{p,x}(t)=x$ for some $t \in (0, +\infty]$, and by $\dot{\gamma}_{p,x}(s)$ the tangent vector of $\gamma_{p,x}$ at s. Given $\delta>0$, let $C(p, x, \delta)$ be the set $\{Q \in \bar{M} : \measuredangle_p(\dot{\gamma}_{p,x}(0), \dot{\gamma}_{p,Q}(0))<\delta\}$, where $\measuredangle_p(v, w)$ is the angle between v and w in the tangent space at p. For simplicity, we put $Q(t)=\gamma_{0,Q}(t)$, $C(Q, t)=C(Q(t), Q, \pi/4)$ and $\Delta(Q, t)=C(Q, t) \cap S(\infty)$, for $Q \in S(\infty)$. We call $\Delta(Q, t)$ a surface ball.

Let Δ_M be the Laplace-Beltrami operator on M. A function f on M is harmonic in $D \subset M$, by definition, if $\Delta_M f(x)=0$, $x \in D$. For $x \in M$, let $d\omega^x$ be the harmonic measure relative to x and M, and put $d\omega=d\omega^o$. If $f \in L^p$ $(=L^p(S(\infty), d\omega))$ $(1 \leq p \leq \infty)$, then we denote by $\tilde{f}$ the harmonic extension of f, i.e. $\tilde{f}(x)=\int f(Q)\, d\omega^x(Q)$ when $x \in M$, and $\tilde{f}(x)=f(x)$ when $x \in S(\infty)$. Let $N(f)$ be the nontangential maximal function of f, that is, $N(f)(Q)=\sup\{|\tilde{f}(z)| : z \in \Gamma(Q)\}$, $Q \in S(\infty)$, where $\Gamma(Q)=\{x \in M : Q \in C(x, \gamma_{o,x}(+\infty), \pi/4) \cap S(\infty)\}$. The set $\Gamma(Q)$ is an analogue of Stoltz domains. Hardy spaces on $S(\infty)$ are defined by $H^p=\{f \in L^1 : \|f\|_{H^p}=\|N(f)\|_p<+\infty\}$, $0<p \leq \infty$, where $\|g\|_p=(\int |g|^p d\omega)^{1/p}$, for every measurable function g on $S(\infty)$.

From a modification of the proof of [2, Theorem 7.3], it follows that $H^p=L^p$, $1<p \leq \infty$, but, in general, H^1 is a proper subspace of L^1. C. Fefferman's duality theorem asserts that the dual space of H^1 on $\boldsymbol{R}^n$ is the space BMO. In our context, the space BMO is defined as follows: For $f \in L^1$, let $\|f\|_*=\sup\left\{\frac{1}{\omega(\Delta)}\int_\Delta \left|f(q)-\frac{1}{\omega(\Delta)}\int_\Delta f\,d\omega\right| d\omega(q) : \Delta \text{ is a surface ball}\right\}$ and BMO $=\{f \in L^1 : \|f\|_*<+\infty\}$.

One of our main results is the following: