65. *On Representations of Lie Superalgebras*

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In this note we give a method of constructing irreducible (unitary) representations of a Lie superalgebra $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1$, using (unitary) representations of a usual Lie algebra $\mathfrak{g}_0$, the even part of $\mathfrak{g}$. We propose some problems about this method of extension. If we can solve these problems, the classification of irreducible representations will be achieved. We analyze this method of extension and give some examples.

1. Unitary representations. Let (π, V) be an irreducible representation of $\mathfrak{g}$ on a Z_2-graded complex vector space $V=V_0+V_1$ in the sense of Kac [4, § 1]. On the even part V_0 and also on the odd part V_1 of V, we have naturally representations of the even part $\mathfrak{g}_0$, of which π is called an extension. We call π *unitary* if V is equipped with a positive definite inner product $\langle\cdot,\cdot\rangle$ in V satisfying

(i) $V_0 \perp V_1$ (orthogonal) under $\langle\cdot,\cdot\rangle$, and

(ii) $\langle\cdot,\cdot\rangle$ is $\mathfrak{g}$-invariant in the sense that

$$\langle i\pi(X)v, v'\rangle=\langle v, i\pi(X)v'\rangle \qquad (v, v' \in V,\ X \in \mathfrak{g}_0),$$
$$\langle j\pi(\xi)v, v'\rangle=\langle v, j\pi(\xi)v'\rangle \qquad (v, v' \in V,\ \xi \in \mathfrak{g}_1),$$

where $i=\sqrt{-1}$ and j is a fixed forth root (depending only on π) of -1, i.e., $j^2=\varepsilon i$ with $\varepsilon=\pm 1$. We call j^2 the associated constant for π since the essential thing is not j itself but $j^2=\varepsilon i$. In this case, both $\pi(\mathfrak{g}_0)|V_0$ and $\pi(\mathfrak{g}_0)|V_1$ are usual unitary representations of $\mathfrak{g}_0$.

2. Extension problems. To classify and to construct all the irreducible (unitary) representations of $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1$, we wish to utilize rich results on representations of usual Lie algebra $\mathfrak{g}_0$. Therefore we propose some problems from this point of view.

Problem 1. Take an *irreducible* representation ρ of $\mathfrak{g}_0$ on a complex vector space V_0. Then, do there exist any irreducible representations (π, V) of $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1$ extending (ρ, V_0)? (More exactly, V_0 is imbedded into V as its subspace of degree 0, and ρ is equivalent to $\pi(\mathfrak{g}_0)|V_0$ under this embedding.) If they do exist, construct all of them.

Problem 2. Let (ρ, V_0) be an *irreducible unitary* $\mathfrak{g}_0$-module. Then do there exist any irreducible unitary extensions of (ρ, V_0) to $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1$? If any, in which different ways can we extend it?

After some study, we recognize that the even part (ρ, V_0) of an irreducible representation (π, V) is not irreducible in many cases. As a matter of fact, the adjoint representation of a Lie superalgebra of type A is in such a case. So we generalize the above problems to Problem 1 bis (or