60. *On the Value of the Dedekind Sum*

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Let p and q be relatively prime positive integers. The n^{th} *Dedekind sum for* p, q will be defined by

$$S_n(p, q)=\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^n \qquad (n=1, 2, \cdots),$$

where $[x]$ denotes, as usual, the greatest integer not exceeding x. It is easy to see that $S_1(p, q)=S_1(q, p)=\frac{1}{2}(p-1)(q-1)$ and the following *reciprocity formulas* are known:

$$(1) \qquad \frac{1}{p}S_2(p, q)+\frac{1}{q}S_2(q, p)=\frac{1}{6pq}(p-1)(2p-1)(q-1)(2q-1),$$

$$(2) \qquad \frac{1}{p(p-1)}S_3(p, q)+\frac{1}{q(q-1)}S_3(q, p)=\frac{1}{4pq}(p-1)(q-1)(2pq-p-q+1)$$

(see, for example, Carlitz [3]).

Assume now $p>q$ throughout this paper. One of the methods to prove these reciprocity formulas is to put $[hq/p]=i-1$ $(i=1, 2, \cdots, q)$ and change $S_n(q, p)$ to the sum with respect to i taking the multiplicities of i's into account. Here the multiplicity of i means the number of h which yields the same value of i and is determined as follows: If h ranges from $[(i-1)p/q]+1$ to $[ip/q]$ for $i<q$, then the value of $[hq/p]$ is $i-1$; for $i=q$, however, h ranges only from $[(q-1)p/q]+1$ to $p-1$. (See, for example, Rademacher and Whiteman [6], (3.5).) Therefore, to obtain the reciprocity relation, we have only to apply the equation

$$(3) \qquad \left[\frac{(h+1)q}{p}\right]-\left[\frac{hq}{p}\right]=\begin{cases}1 & \text{if } h=[ip/q]\ (i=1, \cdots, q-1) \text{ or } p-1,\\ 0 & \text{otherwise.}\end{cases}$$

We have now the following lemma.

Lemma. *Put* $r_1=p-[p/q]q$, *then we get the equation*

$$(4) \qquad \left[\frac{(k+1)p}{q}\right]-\left[\frac{kp}{q}\right]=\begin{cases}[p/q]+1 & \textit{if } k=[jq/r_1]\ (j=1, \cdots, r_1-1) \textit{ or } q-1,\\ [p/q] & \textit{otherwise.}\end{cases}$$

Proof. Substituting $p=[p/q]q+r_1$, we get

$$\left[\frac{(k+1)p}{q}\right]-\left[\frac{kp}{q}\right]=\left[\frac{(k+1)r_1}{q}\right]-\left[\frac{kr_1}{q}\right]+\left[\frac{p}{q}\right].$$

Since q and r_1 are relatively prime and $r_1<q$, the equation (4) follows from the equation (3). □

The equation (4) can be used for reducing the Dedekind sum to a sum of fewer terms and thus for giving an algorithm to evaluate the Dedekind sum in some cases.