58. *On a Problem of R. Brauer on Zeta-Functions of Algebraic Number Fields. II*

By Ken-ichi SATO
Faculty of Engineering, Nihon University

(Communicated by Shokichi IYANAGA, M. J. A., June 9, 1987)

1. Let K_1, K_2 be algebraic number fields of finite degrees. Put $K=K_1K_2$, $k=K_1\cap K_2$ and consider the following quotient of Dedekind zeta-functions:

$$\zeta_{K_1,K_2}(s)=\zeta_K(s)\cdot\zeta_k(s)/\zeta_{K_1}(s)\cdot\zeta_{K_2}(s).$$

It was shown by R. Brauer [1] that $\zeta_{K_1,K_2}(s)$ is an entire function of s, if K_1/k and K_2/k are normal. In our previous note [2], we called *R. Brauer's problem* the question asking for other cases in which $\zeta_{K_1,K_2}(s)$ becomes entire. We proved that this takes place in the following cases:

(i) $K_1=Q(\sqrt[p]{a})$, $K_2=Q(\sqrt[p]{b})$, where p is an odd prime and a, b are relatively prime p-free integers $\neq 1$.

(ii) $K_1=Q(\sqrt[p]{a})$, $K_2=Q(\sqrt[q]{b})$ where p, q are distinct odd primes and a, b are relatively prime, respectively p-free and q-free integers $\neq 1$.

In the present note, we shall show that these results can be derived in a generalized form from a theorem on "supersolvable extensions" as stated below. The letters k, K, L, M (sometimes with suffixes) will denote throughout this note algebraic number fields of finite degrees.

2. If K/k is normal and Gal(K/k) is supersolvable, K/k itself will be called *supersolvable*. Then there exists a chain of intermediate fields $K=k_\nu\supset k_{\nu-1}\supset\cdots\supset k_0=k$ such that all k_i/k are normal and $k_i\supset k_{i-1}$ are cyclic, $i=\nu, \nu-1, \cdots, 1$. It is known that if K/k is supersolvable, the Artin L-function $L(s, \chi, K/k)$ for every non-principal character χ of Gal (K/k) is entire (cf. [3]).

Theorem. *Let $K=K_1K_2$, $k=K_1\cap K_2$. Let M/k, M_1/k be galois closures of K/k, K_1/k respectively. If M/k is supersolvable and $M_1\cap K_2=k$, then $\zeta_{K_1,K_2}(s)$ is entire.*

Proof. Put $G=\mathrm{Gal}(M/k)$, $G_1=\mathrm{Gal}(M_1/k)$, $H_1=\mathrm{Gal}(M_1/K_1)$. Then we have after Artin $\zeta_{K_1}(s)=L(s, 1_{H_1}, M_1/K_1)=L(s, 1_{H_1}^{G_1}, M_1/k)$, where 1_{H_1} is the principal character of H_1 and $1_{H_1}^{G_1}$ the same character induced to G_1. Likewise $\zeta_k(s)=L(s, 1_{G_1}, M_1/k)$. Now we can write $1_{H_1}^{G_1}=1_{G_1}+\sum_i\lambda_i$, where λ_i are nonprincipal irreducible characters of G_1, so that we obtain

(1) $\zeta_{K_1}(s)/\zeta_k(s)=\prod_i L(s, \lambda_i, M_1/k)=\prod_i L(s, \tilde{\lambda}_i, M/k)$. Here $\tilde{\lambda}_i$ is the character λ_i lifted to Gal(M/k). We give the following diagram for the sake of convenience.