7. *Commutator Relations in Kac-Moody Groups*

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In this note, we will calculate the commutator relations in Kac-Moody groups over commutative rings. The commutator relations have been discussed already in Tits [4]. Our approach is more elementary and more explicit.

1. Chevalley systems. Let A be an $n\times n$ generalized Cartan matrix, $\mathfrak{g}$ the associated Kac-Moody algebra over $\boldsymbol{C}$, being generated by the Cartan subalgebra $\mathfrak{h}$ and the Chevalley generators $e_1,\cdots,e_n, f_1,\cdots,f_n$, and Δ the root system of $(\mathfrak{g},\mathfrak{h})$ with the standard fundamental system $\Pi=\{\alpha_1,\cdots,\alpha_n\}$. Then we obtain the root space decomposition $\mathfrak{g}=\oplus_{\alpha\in\Delta}\mathfrak{g}^\alpha$ with $\mathfrak{g}^0=\mathfrak{h}$, $\mathfrak{g}^{\alpha_i}=\boldsymbol{C}e_i$ and $\mathfrak{g}^{-\alpha_i}=\boldsymbol{C}f_i$ $(1\le i\le n)$. The Chevalley involution ω is defined to be the involutive automorphism of $\mathfrak{g}$ given by $\omega(e_i)=-f_i$, $\omega(f_i)=-e_i$, $\omega(h)=-h$ for all $1\le i\le n$ and $h\in\mathfrak{h}$. By the definition, $\omega(\mathfrak{g}^\alpha)=\mathfrak{g}^{-\alpha}$ for all $\alpha\in\Delta$ (cf. [3]).

For each $\alpha\in\Delta^{re}$, the set of real roots, a pair $(e_\alpha, e_{-\alpha})\in\mathfrak{g}^\alpha\times\mathfrak{g}^{-\alpha}$ is called a Chevalley pair for α if $[e_\alpha,e_{-\alpha}]=h_\alpha$ and $\omega(e_\alpha)+e_{-\alpha}=0$, where h_α is the coroot of α. There are precisely two Chevalley pairs for each $\alpha\in\Delta^{re}$. If one is $(e_\alpha,e_{-\alpha})$, then $(-e_\alpha,-e_{-\alpha})$ is the other. We choose and fix a Chevalley pair for each positive real root α with $e_{\alpha_i}=e_i$, $e_{-\alpha_i}=f_i$ $(1\le i\le n)$. Then the set $C=\{e_\alpha\mid\alpha\in\Delta^{re}\}$ is called a Chevalley system for Δ^{re}. Notice that $C\cup\{h_{\alpha_1},\cdots,h_{\alpha_n}\}$ is a Chevalley basis of $\mathfrak{g}$ if A is of finite type (cf. [1], [2]).

Let $\alpha,\beta\in\Delta^{re}$. If $\alpha+\beta\in\Delta^{re}$, then we define the number $N_{\alpha\beta}$ by $[e_\alpha,e_\beta]=N_{\alpha\beta}e_{\alpha+\beta}$. Then we obtain the following result, which is useful for computing the commutator relations.

Theorem 1. *Let $\alpha,\beta\in\Delta^{re}$ with $\alpha+\beta\in\Delta^{re}$, and let $\beta-p\alpha,\cdots,\beta,\cdots,\beta+q\alpha$ $(p,q\in\boldsymbol{Z}_{\ge 0})$ be the α-string through β. Then $N_{\alpha\beta}=\pm(p+1)$. In particular, $N_{\alpha\beta}\in\boldsymbol{Z}$.*

Proof. We can assume $n=2$, hence A is symmetrizable. We fix a symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}^*$ induced by A and having the property $(\alpha_i,\alpha_i)>0$. Then we see

$$N_{\alpha\beta}^2=(p+1)\left\{(p+1)-\left(2\frac{(\beta,\alpha)}{(\alpha,\alpha)}+1\right)\left(1-q\frac{(\alpha,\alpha)}{(\beta,\beta)}\right)\right\}$$

(cf. [2]). If $(\alpha,\alpha)\ge(\beta,\beta)$ and $(\beta,\alpha)<0$, then $\beta+i\alpha\in\Delta^{re}$ $(-p\le i\le q)$, and $p=0,1$, hence $2(\beta,\alpha)/(\alpha,\alpha)=-1$. If $(\alpha,\alpha)\ge(\beta,\beta)$ and $(\beta,\alpha)\ge 0$, then $q=1$ and $(\alpha+\beta,\alpha+\beta)>(\alpha,\alpha)=(\beta,\beta)$. If $(\alpha,\alpha)<(\beta,\beta)$, then $(\alpha,\beta)<0$ and $p=0$, hence $(\beta,\beta)/(\alpha,\alpha)=q$. Therefore, in any case, we obtain $N_{\alpha\beta}^2=(p+1)^2$ and $N_{\alpha\beta}=\pm(p+1)$. □

2. Commutator relations. Let $G(R)$ be a Kac-Moody group over a