56. *On Uniform Distribution of Sequences*

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Let z: $z_0=0<z_1<z_2\cdots$ be a subdivision of the interval $[0,\infty)$ with $z_n\to\infty$ as $n\to\infty$. For an increasing sequence $(x_n)_{n=1}^{\infty}$ of non-negative real numbers, define the sequence (i_n) of positive integers by

$$z_{i_n-1}\leqq x_n<z_{i_n}.$$

Then (x_n) is said to be uniformly distributed modulo the subdivision z if the sequence

$$\{x_n\}_z=\frac{x_n-z_{i_n-1}}{z_{i_n}-z_{i_n-1}} \tag{1}$$

is uniformly distributed mod 1, i.e., if

$$\lim_{n\to\infty}(1/N)A(x,N,\{x_n\}_z)=x \qquad (0\leqq x\leqq 1), \tag{2}$$

where $A(x,N,\{x_n\}_z)$ denotes the number of indices n, $1\leqq n\leqq N$ such that $\{x_n\}_z$ is less than x.

The following distribution properties of the sequence $(x_n)=(n\theta)$ (θ an arbitrary positive real number) are well-known:

(i) If $z_n-z_{n-1}\to\infty$ and $z_n/z_{n-1}\to 1$ as $n\to\infty$, then (x_n) is uniformly distributed mod z (W. J. Le Veque [6]).

(ii) If z_n-z_{n-1} is decreasing, then (x_n) is uniformly distributed mod z for almost all θ; this result also holds in the case $(x_n)=(n^{\gamma}\theta)$ for any fixed $\gamma>0$ (H. Davenport and W. J. Le Veque [3]).

(iii) If $z_n/z_{n-1}\to 1$ as $n\to\infty$ and if the number of terms z_n with $z_n\leqq N$ is less than $c\cdot N^{2-\delta}$ $(c,\delta>0)$, then (x_n) is uniformly distributed mod z for almost all θ (H. Davenport and P. Erdös [2]).

In the following we prove a generalization of some of these results by an elementary method (cf. [7]). For this purpose we define a sequence (x_n) to be *almost uniformly distributed* mod z if there is an infinite sequence $N_1<N_2<\cdots$ of positive integers such that

$$\lim_{n\to\infty}(1/N_i)A(x,N_i,\{x_n\}_z)=x \qquad (0\leqq x\leqq 1); \tag{3}$$

see Definitions 1.2 and 7.2 in the monograph of L. Kuipers and H. Niederreiter [5]. For further results on uniform distribution modulo a subdivision see Burkhard [1], P. Kiss [4].

Theorem. *Let θ be a positive real number and let $z=(z_n)$ be an increasing sequence of real numbers with conditions $z_0=0$ and $z_n/n\to\infty$ as $n\to\infty$. Then the sequence $(x_n)=(\theta n)$ $(n=1,2,\cdots)$ is almost uniformly distributed*

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