53. Whittaker Models for Highest Weight Representations of Semisimple Lie Groups and Embeddings into the Principal Series

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Let G be a connected, real simple linear Lie group and K its maximal compact subgroup. Assume that G/K is a hermitian symmetric space. The aim of this note is to describe embeddings of irreducible highest weight G-modules, including the holomorphic discrete series and finite-dimensional representations, into two types of interesting induced representations: Kawanaka's generalized Gelfand-Graev representations (GGGRs) and the principal series.

1. Method and preparation. We employ here the method of highest weight vectors (cf. Hashizume [2]). Precisely, we determine all the K-finite highest weight vectors in GGGRs and the principal series by solving systems of differential equations characterizing such vectors. This enables us to describe embeddings of highest weight modules.

We prepare a refined structure theorem for $\mathfrak{g}\equiv\mathrm{Lie}(G)$, due to Moore. Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ with $\mathfrak{k}\equiv\mathrm{Lie}(K)$, be a Cartan decomposition of $\mathfrak{g}$, and θ the corresponding Cartan involution of G. Then there exists a unique central element Z_0 of $\mathfrak{k}$ such that $\mathrm{ad}(Z_0)|\mathfrak{p}$ gives the $\mathrm{Ad}(K)$-invariant complex structure on $\mathfrak{p}$ coming from the given G-invariant one on G/K. Putting $\mathfrak{p}_\pm=\{X\in\mathfrak{p}_C;\ [Z_0,X]=\pm\sqrt{-1}X\}$, one gets a decomposition $\mathfrak{g}_C=\mathfrak{p}_-\oplus\mathfrak{k}_C\oplus\mathfrak{p}_+$. Let $\mathfrak{t}\subseteqq\mathfrak{k}$ be a compact Cartan subalgebra of $\mathfrak{g}$, and Δ the root system of $(\mathfrak{g}_C,\mathfrak{t}_C)$. Select a positive system Δ^+ of Δ such as $\gamma(Z_0)=\sqrt{-1}$ for all non-compact positive roots γ. We denote by $\Delta_{\mathfrak{k}}^+$ (resp. $\Delta_{\mathfrak{p}}^+$) the set of compact (resp. non-compact) positive roots. Construct a sequence $(\gamma_1,\gamma_2,\cdots,\gamma_l)$ of non-compact positive roots inductively as follows: γ_k is the largest root of $\Delta_{\mathfrak{p}}^+$ strongly orthogonal to $\gamma_m: \gamma_k\pm\gamma_m\notin\Delta\cup\{0\}$, for all $m>k$. We take a root vector X_γ for a $\gamma\in\Delta$ satisfying

$$X_\gamma-X_{-\gamma},\quad \sqrt{-1}(X_\gamma+X_{-\gamma})\in\mathfrak{k}+\sqrt{-1}\mathfrak{p},\quad [X_\gamma,X_{-\gamma}]=H'_\gamma.$$

Here, $H'_\gamma\equiv 2H_\gamma/\gamma(H_\gamma)$ with the element $H_\gamma\in\sqrt{-1}\mathfrak{t}$ determined by $\gamma(H)=B(H,H_\gamma)$ $(H\in\mathfrak{t}_C)$, through the Killing form B of $\mathfrak{g}_C$.

We put $H_k=X_{\gamma_k}+X_{-\gamma_k}\in\mathfrak{p}$ for $1\leqq k\leqq l$. Then, $\mathfrak{a}_p\equiv\sum_{1\leqq k\leqq l}\boldsymbol{R}H_k$ is a maximal abelian subspace of $\mathfrak{p}$. Let μ be a Cayley transform of $\mathfrak{g}_C$ defined by $\mu=\exp((\pi/4)\cdot\sum_{1\leqq k\leqq l}\mathrm{ad}(X_{\gamma_k}-X_{-\gamma_k}))$. Then, $\mu(H_k)=H'_{\gamma_k}$, whence $\psi_k\equiv(\gamma_k/2)\circ(\mu|\mathfrak{a}_p)$ $(1\leqq k\leqq l)$ form an orthogonal basis of $\mathfrak{a}_p^*$, the dual space of $\mathfrak{a}_p$. The root system Ψ of $(\mathfrak{g},\mathfrak{a}_p)$ is related with Δ via $(\Delta\circ(\mu|\mathfrak{a}_p))\cup\{0\}=\Psi\cup\{0\}$. We select a positive system Ψ^+ of Ψ consistent with $\Delta^+\subseteqq\Delta$ under this rela-