# 40. *Isolated Singularities and Positive Solutions of Elliptic Equations in $R^n$*

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**1. Introduction.** We consider positive solutions to a second-order locally uniformly elliptic equation

(1) $$Pu\equiv[-\sum_{j,k=1}^{n}\partial_j(a_{jk}\partial_k)+\sum_{j=1}^{n}b_j\partial_j+c]u=0$$

in a domain $D$ of $R^n$, where $n\geqq 2$, $\partial_j=\partial/\partial x_j$, and $a_{jk}$, $b_j$, $c$ are real-valued functions in $L_{\infty,\mathrm{loc}}(D)$, $L_{2p,\mathrm{loc}}(D)$, $L_{p,\mathrm{loc}}(D)$, respectively, for some $p>n/2$. A positive solution means a positive continuous function belonging locally to the Sobolev space of order 1 (i.e., in $H^1_{\mathrm{loc}}(D)$) and satisfying (1) in the weak sense (cf. [8]). We say that $(P, D)$ is *subcritical* if for some $y$ in $D$ there exists a positive Green's function $G(\cdot, y)$ for $P$ in $D$: $G\in L_{1,\mathrm{loc}}(D)$, $G>0$, and $P_xG(x, y)=\delta(x-y)$ in the weak sense, where $\delta$ is Dirac's distribution (cf. [8]). Note that the subcriticality of $(P, D)$ implies the existence of a positive solution to (1) in $D$ (cf. [1], [2], and [3]).

The purpose of this note is to establish a relationship between positive solutions to an elliptic equation in a punctured ball and those to a corresponding equation in $R^n$. For isolated singularities of positive solutions, see [4, 5, 6, 7, 9]; as for positive solutions in $R^n$, see [1, 2, 3].

Let $\Omega=\{x\in R^n; 0<|x|<R\}$ for some $R>0$, and let $P$ be an elliptic operator of the form as in (1) with $D=\{0<|x|<R+1\}$. Choose a positive continuous function $g$ on $\{0<|x|\leqq R\}$ satisfying the equation $-\sum_{j,k}\partial_j(a_{jk}\partial_k g)+\sum_j b_j\partial_j g=0$ in $\Omega$. Define a generalized Kelvin transformation $K$ by

(2) $$Ku(y)=u(y^*)/g(y^*), \qquad y^*=y|y|^{-2}.$$

Then the same argument as in the proof of Theorem 2 of [7] shows that if $u$ is a solution of $Pu=0$ in $\Omega$, then $Ku$ is a solution of the equation $P'v=0$ in $E=\{y\in R^n; |y|>1/R\}$, where

(3) $$P'=-\sum_{i,l}\partial_i(a'_{il}\partial_l)+\sum_i b'_i\partial_i+c',$$
$$c'(y)=h(y)|y|^{-4}c(y^*), \qquad h(y)=[g(y^*)|y|^{2-n}]^2,$$
$$b'_i(y)=\sum_{j=1}^{n}h(y)|y|^{-2}b_j(y^*)(\delta_{ij}-2y_iy_j/|y|^2),$$
$$a'_{il}(y)=\sum_{j,k=1}^{n}h(y)a_{jk}(y^*)(\delta_{ij}-2y_iy_j/|y|^2)(\delta_{kl}-2y_ky_l/|y|^2).$$

Here $\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ if $i\neq j$. For $K\geqq 0$, let $P^K$ be an elliptic operator in $R^n$ such that $P^K=P'$ in $E$ and $P^K=-\Delta+K\chi_R$ in $R^n\backslash\bar{E}$ with $\chi_R$ being the characteristic function of the set $\{1/2R<|x|<1/R\}$. Let

(4) $$H_+(P^K, R^n)=\{v\in H^1_{\mathrm{loc}}(R^n); P^Kv=0 \text{ and } v>0 \text{ in } R^n\},$$

(5) $$H_+(P, \Omega, \{0\})=\{u\in H^1_{\mathrm{loc}}(\Omega); u \text{ is continuous on } \bar{\Omega}\backslash\{0\} \text{ and vanishes on } \{|x|=R\},\ Pu=0 \text{ and } u>0 \text{ in } \Omega\}$$

be Fréchet spaces equipped with the metrics induced by $L_\infty$-norms on com-