38. *Some Remarks on C-semigroups and Integrated Semigroups*

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1. Introduction. Let X be a Banach space and let $B(X)$ be the set of all bounded linear operators from X into itself. Let C be an injective operator in $B(X)$ and the range $R(C)$ be dense in X. According to Davies and Pang [3], a family $\{S(t)\,;\,t\geq 0\}$ in $B(X)$ is called *C-semigroup*, if

(1.1) $S(t+s)C=S(t)S(s)$ for $t,\ s\geq 0$, and $S(0)=C$,

(1.2) $S(\cdot)x\colon [0,\infty)\to X$ is continuous for $x\in X$,

(1.3) there are $M\geq 0$ and $a\in \boldsymbol{R}\equiv(-\infty,\infty)$ such that $\|S(t)\|\leq Me^{at}$ for $t\geq 0$.

We define an operator G by $Gx=\lim_{t\to 0+}(C^{-1}S(t)x-x)/t$ for $x\in D(G)$ $\equiv\{x\in R(C)\,;\,\lim_{t\to 0+}(C^{-1}S(t)x-x)/t \text{ exists}\}$. It is known that G is densely defined and closable, $\lambda-\overline{G}$ is injective for $\lambda>a$ and

(1.4) $(\lambda-\overline{G})\int_0^\infty e^{-\lambda t}S(t)x\,dt=Cx$ for $x\in X$ and $\lambda>a$.

(See [3], [4].) The closure $\overline{G}$ is called the *C-c.i.g.* of $\{S(t)\,;\,t\geq 0\}$.

Let n be a positive integer. A family $\{U(t)\,;\,t\geq 0\}$ in $B(X)$ is called *n-times integrated semigroup* (see [2]), if

(1.5) $U(\cdot)x\colon [0,\infty)\to X$ is continuous for $x\in X$,

(1.6) $U(t)U(s)x=\dfrac{1}{(n-1)!}\left(\int_t^{s+t}(s+t-r)^{n-1}U(r)x\,dr-\int_0^{s}(s+t-r)^{n-1}U(r)x\,dr\right)$

for $x\in X$ and $s,\ t\geq 0$, and $U(0)=0$,

(1.7) $U(t)x=0$ for all $t>0$ implies $x=0$,

(1.8) there are $M\geq 0$ and $\omega\in\boldsymbol{R}$ such that $\|U(t)\|\leq Me^{\omega t}$ for $t\geq 0$.

For convenience we call a C_0-semigroup also *0-times integrated semigroup.*

It is known [2] that if $\{U(t)\,;\,t\geq 0\}$ is an n-times integrated semigroup, then there exists a unique closed linear operator A such that $(\omega,\infty)\subset\rho(A)$ (the resolvent set of A) and

(1.9) $R(\lambda\,;\,A)x\,(\equiv(\lambda-A)^{-1}x)=\int_0^\infty \lambda^n e^{-\lambda t}U(t)x\,dt$ for $x\in X$ and $\lambda>\omega$.

The operator A is called the *generator* of $\{U(t)\,;\,t\geq 0\}$.

The purpose of this paper is to prove the following theorems.

Theorem 1. *Let A be a densely defined closed linear operator in X with $\rho(A)\neq\emptyset$. Let $c\in\rho(A)$ and $n\geq 0$ be an integer. The following* (i)–(iii) *are equivalent:*

(i) *A is the generator of an n-times integrated semigroup $\{U(t)\,;\,t\geq 0\}$.*

(ii) *A is the C-c.i.g. of a C-semigroup $\{S(t)\,;\,t\geq 0\}$ with $C=R(c\,;\,A)^n$.*