32. *On Eisenstein Series of Degree Two*

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Our motive is to see that the conditional theorem E in Introduction of [5] holds unconditionally for cusp forms. To do it, we study real analytic Eisenstein series with level. q denotes a fixed natural number $\geqq 3$ and we put $\Gamma_n = Sp_n(\boldsymbol{Z})$, $\Gamma_n(q) = \{M \in \Gamma_n \mid M \equiv 1_{2n} \bmod q\}$, $\Gamma_n(\infty) = \left\{\begin{pmatrix} * & * \\ 0^{(n)} & * \end{pmatrix} \in \Gamma_n\right\}$, and $H = \{Z \in M_2(\boldsymbol{C}) \mid Z = {}^tZ,\ \operatorname{Im} Z > 0\}$. If $C, D \in M_2(\boldsymbol{Z})$ satisfy $C{}^tD = D{}^tC$ and (C, D) is primitive, then we write $(C, D) = 1$.

§1. Definition of Eisenstein series and their relations. For $Z \in H$ and $s \in \boldsymbol{C}$ we put

$$E(Z, s) = \sum_M |Y(M\langle Z\rangle)|^s,$$
$$E'(Z, s) = |Y|^s \sum_{C,D} (abs\,|CZ+D|)^{-2s}$$

where M runs over $(\Gamma_2(q) \cap \Gamma_2(\infty))\backslash\Gamma_2(q)$, $Y(*)$ denotes the imaginary part of $*$, and (C, D) runs over $\Gamma_1(q)\backslash\{(C, D) \in M_{2,4}(\boldsymbol{Z}) \mid C{}^tD = D{}^tC,\ (C, D) \equiv (0, 1_2) \bmod q\}$. For $V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}$, $V_i \in M_2(\boldsymbol{Q})$ such that qV, $qVI{}^tV \left(I = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}\right)$ are integral, we put

$$\tilde{E}'(Z, s\,;V) = |Y|^s \sum_{(C,D)} \exp(\pi i \operatorname{tr}((V_4, -V_3){}^t(2C - V_1,\ 2D - V_2)))$$
$$\cdot (abs\,|CZ+D|)^{-2s},$$

where (C, D) runs over $\Gamma_1(q)\backslash\{(C, D) \in M_{2,4}(\boldsymbol{Q}) \mid C{}^tD = D{}^tC,\ (C, D) \equiv (V_1, V_2) \bmod 1,\ rk(C, D) = 2\}$.

Proposition 1. (i) *$E'(Z, s) = (2\#SL_2(\boldsymbol{Z}/q))^{-1} \sum_a \{\sum_U (abs\,|U|)^{-2s}\} \times E(u(a)Z, s)$ where a, U run over $(\boldsymbol{Z}/q\boldsymbol{Z})^\times/\{\pm 1\}$, $\Gamma_1(q)\backslash\{U \in M_2(\boldsymbol{Z}) \mid a|U| \equiv \pm 1 \bmod q\}$ respectively and $u(a)$ is an element of Γ_2 such that $u(a) \equiv diag(1, a^{-1}, 1, a) \bmod q$.*

(ii) *$E(Z, s) = 2\varphi(q)^{-1} \sum_a (\sum_\chi \chi(a) L(2s, \chi)^{-1} L(2s-1, \chi)^{-1}) E'(u(a)Z, s)$, where a runs over the same set as above and χ runs over even Dirichlet characters modulo q.*

(iii) *For $V = q^{-1}\begin{pmatrix} 0 & a1_2 \\ 0 & cJ \end{pmatrix} \in M_4(\boldsymbol{Q})$ where a, c are relatively prime integers and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \Gamma_1$, $\tilde{E}'(Z, s\,;V)$ is a linear combination of $E(\sigma Z, s)(\sigma \in \Gamma_2)$. If $a = 1$, $c = 0$, then $\tilde{E}'(Z, s\,;V) = q^{4s}E'(Z, s)$.*

§2. Analytic continuation of Eisenstein series. For $Z = {}^tZ \in M_n(\boldsymbol{C})$ with a positive definite imaginary part, put

$$\sum (abs\,|Z+S|)^{-2s} = \sum_h \exp(2\pi i \operatorname{tr}(h \operatorname{Re} Z))\xi_n(\operatorname{Im} Z, h\,; s, s)$$