29. A Characterization of Certain Real Quadratic Fields

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(Communicated by Shokichi IYANAGA, M. J. A., March 12, 1986)

§1. **Introduction.** Let d be a positive square-free integer. We denote by $\omega(d)$ the algebraic integer $\sqrt{d}$ (resp. $(1/2)(1+\sqrt{d})$) in the real quadratic field $\boldsymbol{Q}(\sqrt{d})$ if $d\equiv 2$ or $3 \pmod 4$ (resp. $d\equiv 1 \pmod 4$), and by $\Delta(d)$ and $h(d)$ the discriminant and the class number of $\boldsymbol{Q}(\sqrt{d})$, respectively. The positive real quadratic irrational $\omega(d)$ can be expanded into the periodic infinite continued fraction:

$$\omega(d)=[a_0, \dot{a}_1, \cdots, \dot{a}_k]=[a_0, a_1, \cdots, a_k, a_1, \cdots, a_k, \cdots]$$
$$=a_0+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots,$$

where $a_0, a_1, \cdots$ are positive integers. We call k the *period* of $\omega(d)$ or of $\boldsymbol{Q}(\sqrt{d})$ and denote it by $k(d)$.

The purpose of this note is to give a characterization of real quadratic fields $\boldsymbol{Q}(\sqrt{d})$ with $h(d)=k(d)=1$, in analogy to Rabinovitch's theorem ([5], [6]) characterizing imaginary quadratic fields whose class number is 1.

§2. **Preliminaries.** We recall some facts about integral indefinite binary quadratic forms (cf. [2], Ch. VI). Let $Q(\Delta(d))$ denote the set of integral quadratic forms $aX^2+bXY+cY^2$ with the discriminant $\Delta(d)=b^2-4ac$. Two forms $aX^2+bXY+cY^2$ and $a'X^2+b'XY+c'Y^2$ in $Q(\Delta(d))$ are said to be (*properly*) *equivalent* if $a'(X')^2+b'X'Y'+c'(Y')^2=aX^2+bXY+cY^2$, $(X', Y')=(X, Y)M$, for some $M\in SL_2(\boldsymbol{Z})$. We denote by $Q_+(\Delta(d))$ the quotient of $Q(\Delta(d))$ by this equivalence relation. There is a natural bijection between $Q_+(\Delta(d))$ and the ideal class group of $\boldsymbol{Q}(\sqrt{d})$ in the narrow sense. We shall denote its order by $h_+(d)$.

A quadratic form $aX^2+bXY+cY^2$ in $Q(\Delta(d))$ is said to be *reduced* if $0<\sqrt{\Delta(d)}-b<2|a|<\sqrt{\Delta(d)}+b$. Using the continued fraction $\omega(d)=[a_0, \dot{a}_1, \cdots, \dot{a}_{k(d)}]$, we define reduced forms, in $Q(\Delta(d))$, $\Phi_i=(-1)^i A_i X^2+B_i XY+(-1)^{i+1}A_{i+1}Y^2$, $i=0, 1, \cdots$, where A_i and B_i are inductively defined by $A_0=1$, $B_0=\mathrm{Tr}\,(a_0-\omega(d))$, $A_1=-\mathrm{Nm}\,(a_0-\omega(d))$, $B_{i+1}+B_i=2a_{i+1}A_{i+1}$ and $(B_i+\sqrt{\Delta(d)})/(2A_{i+1})=[a_{i+1}, a_{i+2}, a_{i+3}, \cdots]$. By the periodicity of $\omega(d)$, we get $\Phi_{k(d)}=\Phi_0$ or $\Phi_{2k(d)}=\Phi_0$ according as $k(d)$ is even or odd. Moreover any reduced form which is equivalent to Φ_0 coincides with Φ_i for some i.

§3. **Finiteness of the number of real quadratic fields with given class number and period.** Let $\omega(d)=[a_0, \dot{a}_1, \cdots, \dot{a}_{k(d)}]$ be as above; then we have the following:

Lemma 1. (1) *$a_i=a_{k(d)-i}$ for $0<i<k(d)$ and* $a_{k(d)}=\mathrm{Tr}\,(a_0-\omega(d))$.