27. Path Integral for the Weyl Quantized Relativistic Hamiltonian[†]

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1. Introduction. The aim of this note is to give a path integral representation of the solution of the Cauchy problem for

(1.1) $$\partial_t\psi(t, x) = -[H - mc^2]\psi(t, x), \quad t>0, \quad x \in \boldsymbol{R}^d.$$

Here c is the light velocity. H is the quantum Hamiltonian via the Weyl correspondence, i.e. the pseudo-differential operator ([1], [2], [6])

(1.2) $$(Hg)(x) = (2\pi)^{-d}\iint_{\boldsymbol{R}^d\times\boldsymbol{R}^d} e^{i(x-y)p} h\left(p, \frac{x+y}{2}\right) g(y)dydp, \qquad g \in \mathcal{S}(\boldsymbol{R}^d),$$

associated with the classical Hamiltonian

(1.3) $$h(p, x) = [(cp - eA(x))^2 + m^2c^4]^{1/2} + e\Phi(x), \quad p \in \boldsymbol{R}^d, \quad x \in \boldsymbol{R}^d,$$

of a relativistic spinless particle of mass $m>0$ and charge e interacting with electromagnetic vector and scalar potentials $A(x)$ and $\Phi(x)$ (e.g. [5]). The Planck constant $\hbar$ is taken to equal 1.

The present approach is a rigorous application of the phase space path integral or Hamiltonian path integral method with the "midpoint" prescription ([6], [7]). The path space measure used is a probability measure on the space of the right-continuous paths $X: [0, \infty) \to \boldsymbol{R}^d$ having the left-hand limits. Each path $X(s)$ is called a d-dimensional time homogeneous Lévy process ([3], [4]). The path integral formula obtained has a close analogy with the Feynman-Kac-Itô formula for the quantum Hamiltonian of a nonrelativistic spinless particle of the same mass and charge interacting with vector and scalar potentials (e.g. [8]).

2. Path integral representation. To formulate our result we need some notions from a time homogeneous Lévy process ([3], [4]). The path space measure which we are going to use is the probability measure $\lambda_{0,x}$ on the space $D_{0,x}([0, \infty) \to \boldsymbol{R}^d)$ of the right-continuous paths having the left-hand limits and satisfying $X(0)=x$ whose characteristic function is given by

(2.1) $$\exp\{-t[(c^2p^2 + m^2c^4)^{1/2} - mc^2]\} = \int e^{ip(X(t)-X(0))} d\lambda_{0,x}(X).$$

The Lévy-Khinchin formula turns out to be

(2.2) $$(c^2p^2 + m^2c^4)^{1/2} - mc^2 = -\int_{\boldsymbol{R}^d\setminus\{0\}} [e^{ipy} - 1 - ipyI_{\{|y|<1\}}(y)]n(dy).$$

Here $n(dy)$ is the Lévy measure which is a σ-finite measure on $\boldsymbol{R}^d\setminus\{0\}$ satis-

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