# 3. Propagation of Chaos for the Two Dimensional Navier-Stokes Equation

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In this paper we establish a rigorous derivation of the two dimensional vorticity equation associated with the Navier-Stokes equation from a many particle system as a propagation of chaos.

It is well known that an incompressible and viscous two dimensional fluid, under the action of an external conservative field is described by the following evolution equations

$$\nabla_t v(t, z)+(u\cdot\nabla)v(t, z)-\nu\Delta v(t, z)=0, \tag{1}$$

$$\begin{cases} v(t, z)=\mathrm{curl}\, u(t, z)=\nabla_x u_2-\nabla_y u_1, \\ \nabla\cdot u=0, \qquad z=(x, y)\in R^n \end{cases} \tag{2}$$

where $u=(u_1, u_2)\in R^2$ is the velocity field and $\nabla_x=\partial/\partial x$, $\nabla_y=\partial/\partial y$, $\nabla=(\nabla_x, \nabla_y)$. $\nu>0$ denotes the viscosity constant. Introducing the operator $\nabla^\perp=(\nabla_y, -\nabla_x)$, by virtue of $\nabla\cdot u=0$, one obtains

$$u(t, z)=\int_{R^2}(\nabla^\perp G)(z-z')v(t, z')dz', \tag{3}$$

where $G(z)=-(2\pi)^{-1}\log|z|$ is the fundamental solution of the Poisson equation. By means of (3), (1) turns to be a closed equation and is nothing but a McKean's type non-linear equation (see H. P. McKean [1]). Hence a probabilistic treatment for the equation (1) is possible. Such an observation for the two dimensional Navier-Stokes equation was made by Marchioro-Pulvirenti in [2]. We shall discuss "a propagation of chaos for the equation (1)".

Let $\{Z_t\}$ denote the McKean process associated with (1);

$$dZ_t=\sigma dB_t+E[(\nabla^\perp G)(Z_t-Z'_t)\,|\,Z_t], \qquad \sigma=\sqrt{2\pi} \tag{4}$$

where $B_{\cdot}$ is a 2-dimensional Brownian motion and $Z'_{\cdot}$ is an independent copy of $Z_{\cdot}$.

The $n$ particle system associated with (1) are described by the following S.D.E.s,

$$dZ_t^i=\sigma dB_t^i+(n-1)^{-1}\sum_{\substack{j\neq i\\ j=1}}^{n}(\nabla^\perp G)(Z_t^i-Z_t^j)dt, \qquad 1\leqq i\leqq n, \tag{5}$$

where $(B_{\cdot}^1, \cdots, B_{\cdot}^n)$ is a $2n$-dimensional Brownian motion. Since the coefficients of (4) have singularities at $\mathfrak{N}=\bigcup_{i\neq j}\{z=(z_1, \cdots, z_n)\in R^n,\ z_i\neq z_j\}$, it is not trivial to see that the solution of (4) defines a conservative diffusion process on $R^{2n}$. However, if it starts out side of $\mathfrak{N}$, it can be shown that this diffusion process does not hit $\mathfrak{N}$ (see Osada [4]).