22. *Local Analytic Dimensions of a Subanalytic Set*

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The class of subanalytic sets in a real-analytic manifold M is, by definition, generated by the images of proper real-analytic maps into M with respect to the elementary set-theoretical operations, i.e., finite union, finite intersection and difference. A subanalytic set X in M admits a locally finite stratification in which strata are locally closed real-analytic submanifolds (smooth and connected) of M, say X_i, and are subanalytic themselves in M. (cf. [3]) This enables us to define the topological dimension of X at each point x of M as follows:

$$t\text{-}\dim_x X = \max\{\dim X_i : x \in \bar{X}_i\}$$

which is independent of the choice of stratification.

This article is concerned with other kinds of local dimension of X. First of all, it is known that the closure $\bar{X}$ is also subanalytic in M and is in fact the image of a proper real-analytic map, say $f: Y \to M$. Here we assume that Y is a reduced real-analytic space because the reduction (killing the nilpotents in the structure sheaf of functions) does not affect the image set. Then, for each point $x \in \bar{X}$, we let

$$\boldsymbol{A}_x(X) = \{h \in \mathcal{O}_{M,x} : (h \circ f)_y = 0 \text{ for all } y \in f^{-1}(x)\}$$
$$\boldsymbol{F}_x(X) = \{\hat{h} \in \hat{\mathcal{O}}_{M,x} : (\hat{h} \circ \hat{f})_y = 0 \text{ for all } y \in f^{-1}(x)\}$$

where $\mathcal{O}_{M,x}$ denotes the ring of germs of analytic functions at x on M, $(\ )_y$ does the germ at y, $\hat{\mathcal{O}}_{M,x}$ does the formal completion of $\mathcal{O}_{M,x}$ and $(\ \circ \hat{f})_y$ does the completion map of $(\ \circ f)_y : \mathcal{O}_{M,x} \to \mathcal{O}_{Y,y}$.

Definition. The Krull dimension of $\mathcal{O}_{M,x}/\boldsymbol{A}_x(X)$ is called the analytic dimension of X at x, denoted by $a\text{-}\dim_x X$, while that of $\hat{\mathcal{O}}_{M,x}/\boldsymbol{F}_x(X)$ is called the formal dimension of X at x, denoted by $f\text{-}\dim_x X$.

The dimensions, analytic and formal, defined as above are in fact independent of the choice of f and depend only on the image $\bar{X}$. So are the ideals $\boldsymbol{A}_x(X)$ and $\boldsymbol{F}_x(X)$. Obviously $t\text{-}\dim_x X \le f\text{-}\dim_x X \le a\text{-}\dim_x X$ and the strict inequalities are possible. (cf. [2] and [3])

The result of this article is

Theorem. *Let X be any subanalytic subset of M. Then there exists a locally finite stratification, say $X = \bigcup_i X_i$, with strata X_i all subanalytic in M, having the following properties: For a sufficiently small open neighborhood U_i of X_i in M for each i, we have*

1) *X_i is a closed real-analytic submanifold of U_i*

2) *there exists a coherent ideal sheaf A_i in $\mathcal{O}_{U_i}$ having stalks $A_{ix} = \boldsymbol{A}_x(X)$ for all $x \in X_i$*

3) *$\hat{U}_i$ denoting the formal completion of U_i with respect to the powers*