2. Aitken-Steffensen Acceleration and a New Addition Formula for Fibonacci Numbers[†]

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Interesting addition formulae for Fibonacci numbers and for $\cot(x)$ are discovered by the third author through various numerical observations. It is verified rigorously for the first time by the first author by mathematical induction and then is proved more directly by the use of detailed properties of Cauchy matrix by the second author [3]. We will discuss these formulae in details elsewhere ([1] and [2]).

1. Addition formula. Consider the three parameter family of functions:

$$p(x)=p(\alpha,\beta,\gamma;x)=(\alpha\gamma^x+\beta)/(\gamma^x-1), \tag{1.1}$$

with $\alpha+\beta\neq 0$, $\gamma\neq 0$. For $\alpha=(\sqrt{5}-1)/2$,

$$p(n)=p(\alpha,\alpha^{-1},-\alpha^{-2};n)=F_{n-1}/F_n,\qquad n=1,2,3,\cdots \tag{1.2}$$

is the ratio of the consecutive Fibonacci numbers F_{n-1} and F_n:

$$F_0=0,\quad F_1=1,\quad F_{n+1}=F_n+F_{n-1},\quad n=1,2,3,\cdots. \tag{1.3}$$

The other important examples are

$$\coth(x)=p(1,1,e^2;x) \tag{1.4}$$

and

$$\cot(x)=p(\sqrt{-1},\sqrt{-1},\exp(2\sqrt{-1});x).$$

We state the main theorem:

Theorem 1.1. *For $m=2,3,4,\cdots$ and for any set $(x_1,\cdots,x_m,y_1,\cdots,y_m)$ of arbitrary complex numbers the function $p(x)$ given by (1.1) satisfies:*

$$\begin{aligned} p(x_1+\cdots+x_m+y_1+\cdots+y_m)&=1/{}^t1_m(p(x_i+y_j))^{-1}1_m \\ &=-\det(p(x_i+y_j))/\det\begin{vmatrix} p(x_i+y_j) & 1_m \\ {}^t1_m & 0\end{vmatrix}, \end{aligned} \tag{1.5}$$

provided that $\gamma^{x_i+y_j}\neq 1$ for $1\leq i,j\leq m$. Here $(p(x_i+y_j))$ denotes an $m\times m$ matrix with the indicated (i,j) components, 1_m is the column m-vector with all components 1, ${}^t1_m$ is its transposition.

When applying this formula to the sequences (1.2) we consider the case when x_i and y_j are natural numbers.

By (1.4) we obtain

$$\cot(x_1+\cdots+x_m+y_1+\cdots+y_m)=1/{}^t1_m(\cot(x_i+y_j))^{-1}1_m. \tag{1.6}$$

In particular, putting $x_i=x-iy$ and $y_j=jy$, we have

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