# 15. On Diophantine Properties for Convergence of Formal Solutions

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**1. Introduction.** This paper studies the diophantine properties of the problem of the convergence of formal solutions. Concerning the convergence of formal solutions Kashiwara-Kawai-Sjöstrand studied the equation $Pu\equiv\sum_{|\alpha|=|\beta|\leqq m} a_{\alpha\beta}(x)x^{\alpha}(\partial/\partial x)^{\beta}u(x)=f(x)$ and gave a sufficient condition for the convergence of all formal solutions (cf. [1]). Unfortunately their condition is merely sufficient and not necessary.

As for the necessity there are few works. This is mainly because we must treat rather delicate problems, the diophantine problems. Our object is to study this. We shall introduce the diophantine functions $\sigma_{\xi}$ and $\rho$ which are the generalizations of the Siegel's condition in [4] and the Leray's auxiliary function in [2] respectively. We note that these authors studied similar problems. In terms of these functions we shall give necessary and sufficient conditions for the convergence of formal solutions. We remark that the method here is also applicable to the study of $C^{\infty}$ (or $C^{\omega}$)-hypoellipticity of operators on the torus under slight modifications.

**2. Notations and results.** Let $x=(x_1, x_2)$ be the variable in $\boldsymbol{R}^2$. For $\eta=(\eta_1, \eta_2)\in\boldsymbol{R}^2$ and a multi-index $\alpha=(\alpha_1, \alpha_2)\in\boldsymbol{N}^2$, $\boldsymbol{N}=\{0, 1, 2, \cdots\}$ we set $\eta^{\alpha}=\eta_1^{\alpha_1}\eta_2^{\alpha_2}$ and $(x\cdot\partial)^{\alpha}=(x_1\partial_1)^{\alpha_1}(x_2\partial_2)^{\alpha_2}$ where $\partial=(\partial_1, \partial_2)$ and $\partial_j=\partial/\partial x_j$ $(j=1,2)$. Let $m\geqq 1$ be an integer and let $\omega\in\boldsymbol{C}$. Then we are concerned with the convergence of all formal solutions of the form $u(x)=x^{\omega}\sum_{\eta\in N^2} u_{\eta}x^{\eta}/\eta!$ of the equation with analytic coefficients:

(2.1) $$P(x;\partial)u\equiv\sum_{|\alpha|\leqq m} a_{\alpha}(x)\partial^{\alpha}u(x)=f(x)x^{\omega}$$

where $f(x)$ is a given analytic function. We say that the formal solution $u=x^{\omega}\sum u_{\eta}x^{\eta}/\eta!$ converges if and only if the sum $\sum u_{\eta}x^{\eta}/\eta!$ converges and represents an analytic function of $x$. Let us expand $a_{\alpha}(x)$ into Taylor series, $a_{\alpha}(x)=\sum_{\gamma} a_{\alpha,\gamma}x^{\gamma}/\gamma!$ and define the set $M_P$ by $M_P=\{\gamma-\alpha;\ a_{\alpha,\gamma}\neq 0$ for some $\alpha$ and $\gamma\}$. Then we assume:

(A.1) Every $\eta=(\eta_1, \eta_2)$ in $M_P$ satisfies that $\eta_1+\eta_2\geqq 0$ and, if $\eta_1+\eta_2=0$ then either the condition $\eta_1\geqq 0$ or that $\eta_2\geqq 0$ is satisfied for all $\eta$ in $M_P$.

Roughly speaking this condition implies that Eq. (2.1) is not irregular-singular. We denote by $\Gamma_P$ the smallest convex closed cone with its vertex at the origin which contains $M_P$.

Now let us define

(2.2) $$p(\eta)=\sum_{\alpha,|\alpha|\leqq m} a_{\alpha,\alpha}\eta!/(\eta-\alpha)!\,\alpha!$$