13. On the Connection of the White-Noise and Malliavin Calculi

By Jürgen POTTHOFF*)
Department of Mathematics, Nagoya University
Department of Mathematics, Technical University, Berlin

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0. Introduction. In this note we show how the basics of the Malliavin calculus, see e.g. [5, 6], can be formulated in the frame work of Hida's white-noise calculus [1, 2, 4].

The original motivation of Malliavin to introduce his calculus was to prove statements about the distributions generated by Wiener-functionals, particularly whether these distributions are absolutely continuous. It turns out that his method can be expressed in a rather simple manner by the white-noise calculus. Only basic formulae are needed, such as the chain rule, integration by parts for the ∂_t derivatives and the product rule for ∂_t^*.

Throughout this note we adopt the notation of Kuo [4], however we shall use the definition of the ∂_t-operator given in [3]; for general background see also [1].

1. The chain rule. In this section we establish the chain rule for ∂_t.

Let $(\mathcal{S}'(\boldsymbol{R}), \Sigma, d\mu)$ be white noise and consider a functional φ on $\mathcal{S}'(\boldsymbol{R})$. For fixed $x \in \mathcal{S}'(\boldsymbol{R})$, let φ_x be the functional on $\mathcal{S}(\boldsymbol{R})$ defined by $\varphi_x(\xi)=\varphi(x+\xi)$, $\xi \in \mathcal{S}(\boldsymbol{R})$.

Proposition. *Let $\varphi \in L^p(d\mu)$, $p>1$, so that $\varphi_x(\xi)$ and $\int \varphi_x(\xi)d\mu(x)$ are Fréchet-differentiable on $\mathcal{S}(\boldsymbol{R})$. Then*

$$\frac{\delta}{\delta\xi(t)}\int \varphi_x(\xi)d\mu(x)=\int \frac{\delta}{\delta\xi(t)}\varphi_x(\xi)d\mu(x). \tag{1.1}$$

Corollary.

$$(\partial_t\varphi)(\xi+x)=\left(\frac{\delta}{\delta\xi(t)}\varphi_x\right)(\xi). \tag{1.2}$$

Sketch of proof. (1.1) follows by use of Gâteaux-derivatives and the dominated convergence theorem; the additional use of the chain rule for Fréchet-derivatives gives (1.2).

Lemma (chain rule). *If $\varphi=(\varphi_1, \cdots, \varphi_d)$ is an $\boldsymbol{R}^d$-valued $\mathcal{S}'(\boldsymbol{R})$-functional, with each φ_i satisfying the assumptions of the proposition and $F \in C^1(\boldsymbol{R}^d, \boldsymbol{R})$, so that $F \circ \varphi \in L^q(d\mu)$, $q>1$, then*

$$\partial_t F \circ \varphi=\sum_{i=1}^{d} (F_{,i} \circ \varphi)\partial_t\varphi_i. \tag{1.3}$$

Here $F_{,i}$ denotes the i^{th} partial derivative of F.

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