116. On Surfaces of Class VII_0 with Curves. II

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Introduction. This is a preliminary report on [6]. A compact complex surface is in class VII_0 if it is minimal and if its first Betti number is equal to one. We know many examples of surfaces of class VII_0 (or in short, VII_0 surfaces) with the second Betti number b_2 positive—minimal surfaces with global spherical shells [1]–[3], [7]–[9]. In view of the table [5, (10.3)], it is necessary to study VII_0 surfaces with a cycle of rational curves in detail in order to complete the classification of VII_0 surfaces.

The main consequences to report are as follows. Let S be a VII_0 surface with C a cycle of rational curves. Then $b_2(S)$, the second Betti number of S, is positive, the deformation functor of S is unobstructed and the cycle C is deformed into a nonsingular elliptic curve in a suitable smooth family of deformations of the surface S. An arbitrary deformation of S with a smooth elliptic curve which is a lifting of C is either a blown-up parabolic Inoue surface or generically a blown-up primary Hopf surface (§1, see also [5, (12.3)]). If moreover S has at least b_2 (possibly singular) rational curves, then S has exactly b_2 rational curves and the weighted dual graph of all the curves on S is completely determined. The dual graph turns out to be the same as one of dual graphs of all the curves on minimal surfaces with global spherical shells (§2). As its consequence, Inoue surfaces with b_2 positive are characterized in a uniform manner (§3).

Notations. We use the usual notations in analytic geometry or the same notations as in [5]. In addition to these, we use the following. $[a, b] := \{k \in \boldsymbol{Z}; a \leqq k \leqq b\}$, $L_I := \sum_{i \in I} L_i$, $A \sim B$ iff $c_1(A) = c_1(B)$ in $H^2(S, \boldsymbol{Z})$ for $A, B \in H^1(S, \mathcal{O}_S^*)$.

§1. Smoothing a cycle of rational curves by deformations.

(1.1) **Theorem.** *Let S be a* VII_0 *surface with b_2 positive. Then* $H^2(S, \Theta_S) = 0$.

Proof. Assume the contrary to derive a contradiction. By Serre duality, $H^0(S, \Omega_S^1(K_S)) \neq 0$. Let D be the maximum effective divisor of S such that $H^0(S, \Omega_S^1(K_S - D)) \neq 0$ and let ω be a nonzero element of $H^0(S, \Omega_S^1(K_S - D))$. By definition, zero (ω) is isolated. Then the following is exact

$$0 \longrightarrow \mathcal{O}_S(K_S - D) \xrightarrow{f} \Omega_S^1 \xrightarrow{g} \mathcal{O}_S(2K_S - D)$$

where $f(a) = a\omega$, $g(b) = b \wedge \omega$. Let $\mathcal{H}$ be Coker g. Then supp$(\mathcal{H})$ is isolated points, so that $H^q(S, \mathcal{H}) = 0$ for any $q > 0$. Therefore by taking Euler-Poincaré characteristics, we see

$$b_2 = -\chi(S, \Omega_S^1) = -\chi(S, -K_S + D) - \chi(S, 2K_S - D) + \chi(S, \mathcal{H})$$