115. Casson's Invariant for Homology 3-Spheres and the Mapping Class Group

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1. Introduction. Let Σ_g be a closed orientable surface of genus $g \geqq 2$ and let $\mathcal{M}_g$ be its mapping class group. Let $\mathcal{I}_g$ be the subgroup of $\mathcal{M}_g$ consisting of all mapping classes which act on the homology of Σ_g trivially. It is called the Torelli group of genus g and we have a short exact sequence $1 \to \mathcal{I}_g \to \mathcal{M}_g \to Sp(2g, \boldsymbol{Z}) \to 1$ where $Sp(2g, \boldsymbol{Z})$ is the Siegel modular group. Recently Johnson has obtained several fundamental results concerning the structure of the Torelli group. Among other things he enumerated the Birman-Craggs homomorphisms $\mathcal{I}_g \to \boldsymbol{Z}/2$, which are defined by using the Rohlin invariant of homology 3-spheres (see [1]), and investigated the relationship between them and another abelian quotient of $\mathcal{I}_g$ which he constructed by making use of the action of $\mathcal{I}_g$ on a certain nilpotent quotient of $\pi_1(\Sigma_g)$ (see [3] [4]).

Now the purpose of the present note is to announce our recent related results. Roughly speaking, we have first "lifted" Johnson's result mentioned above in terms of Casson's invariant for homology 3-spheres ([2]) rather than the Rohlin invariant, and then put the computations forward by one step. As a result we can prove the existence of a new method of defining Casson's invariant (see Theorem 9).

2. Johnson's method. Here we briefly recall Johnson's method of investigating the mapping class groups (see [5] for details). For simplicity here we only consider the mapping class group $\mathcal{M}_{g,1}$ of Σ_g relative to an embedded disc $D^2 \subset \Sigma_g$. Write Γ_1 for $\pi_1(\Sigma_g^0)$ $(\Sigma_g^0 = \Sigma_g \setminus \mathring{D}^2)$ and inductively define $\Gamma_{k+1} = [\Gamma_k, \Gamma_1]$ $(k=1, 2, \cdots)$. We may call $N_k = \Gamma_1/\Gamma_k$ the k-th nilpotent quotient of Γ_1. We simply write H for $N_2 = H_1(\Sigma_g, \boldsymbol{Z})$ and choose a symplectic basis $x_1, \cdots, x_g, y_1, \cdots, y_g$ of it. Let $\mathcal{L}$ be the free graded Lie algebra on x_i, y_j and let $\mathcal{L}_k$ be the submodule of $\mathcal{L}$ consisting of all homogeneous elements of degree k. It is a classical result that there exists a natural isomorphism $\Gamma_k/\Gamma_{k+1} \cong \mathcal{L}_k$ (see [7]) and we have a central extension $0 \to \mathcal{L}_k \to N_{k+1} \to N_k \to 1$. We have also natural isomorphisms $\mathcal{L}_2 \cong \wedge^2 H$ and $\mathcal{L}_3 \cong \wedge^2 H \otimes H / \wedge^3 H$. Now $\mathcal{M} = \mathcal{M}_{g,1}$ naturally acts on N_k and set $\mathcal{M}(k)$ to be the subgroup of $\mathcal{M}$ consisting of all elements which act on N_k trivially. $\mathcal{M}(2)$ is nothing but the Torelli group $\mathcal{I}_{g,1}$ and according to [6], $\mathcal{M}(3)$ is the subgroup of $\mathcal{M}$ generated by all Dehn twists on bounding simple closed curves on Σ_g^0. Hereafter we write $\mathcal{K}_{g,1}$ for $\mathcal{M}(3)$. Johnson constructed a homomorphism $\tau_k : \mathcal{M}(k) \to \mathcal{L}_k \otimes H$ such that $\operatorname{Ker} \tau_k = \mathcal{M}(k+1)$ and proved