114. A Note on the Mean Value of the Zeta and L-functions. V

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1. In the previous note of this series we showed an alternative approach to Atkinson's formula. Here we return to the original argument of Atkinson [1], and exploit its ability in the context of the problem dealt by Balasubramanian, Conrey and Heath-Brown [2]. Motivated by Iwaniec [3], they considered the asymptotic evaluation of

$$I(T,A)=\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)A\left(\frac{1}{2}+it\right)\right|^2 dt,$$

where

$$A(s)=\sum_m a(m)m^{-s}$$

and $a(m)$ vanishes for $m>M$. The main term of the integral is

$$T\sum_{k,l}\frac{a(k)\bar{a}(l)}{[k,l]}\left(\log\frac{T(k,l)^2}{2\pi kl}+2\gamma-1\right),$$

and denoting the error-term by $E(T,A)$, they proved, among other things, that

$$E(T,A)\ll T(\log T)^{-B}+M^2T^{\varepsilon}$$

for any fixed B, $\varepsilon>0$ whenever $\log M\ll\log T$, $a(m)\ll m^{\varepsilon}$. Thus $I(T,A)$ is asymptotically equal to the main-term when $M<T^{(1/2)-\varepsilon}$.

Their argument is highly technical, and centers upon a subtle estimation of integrals arising from a Mellin transform of the Γ-factor in the functional equation for $\zeta(s)$. In contrast with this, as we shall show below, a simple modification of Atkinson's argument yields a quite accessible proof of the above as well as the following new estimate:

Theorem.

$$E(T,A)\ll T^{1/3}M^{4/3}T^{\varepsilon}.$$

Remark. (i) Assertions (B) and (C) stated in [2, Theorem 1] can also be proved by refining our argument.

(ii) Our result contains $E(T)\ll T^{1/3+\varepsilon}$.

(iii) The mean square of $E(T,A)$ may be considered. And we stress that in application to the problem of the distribution of the zeros of $\zeta(s)$ as was done in [2] a good mean value estimate of $E(T,A)$ is enough.

(iv) The χ-analogue of our result can be obtained by combining the present note with [4, II].

2. Now we shall show an outline of our argument. The details will be given elsewhere.