113. On Triple L-functions

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We extend the Garrett's result [3] on triple products to different weight case. Details are described in [5]. Let n be a positive integer and $\Gamma_n = Sp(n, \boldsymbol{Z})$. We denote by H_n the Siegel upper half space of degree n. Let $S_k(\Gamma_1)$ be the space of cuspforms of weight k and of degree one. We write Fourier expansion of $f \in S_k(\Gamma_1)$ as $f(z) = \sum_{n=1}^{\infty} a(n, f) e^{2\pi i n z}$. If $f \in S_k(\Gamma_1)$ is a normalized Hecke eigenform and p is a prime, we define semi-simple $M_p(f) \in GL(2, \boldsymbol{C})$ (up to conjugate class) by $\det(1 - tM_p(f)) = 1 - a(p, f)t + p^{k-1}t^2$. For normalized Hecke eigenforms f, g and h, define 'triple L-function' $L(s; f, g, h)$ by

$$L(s; f, g, h) = \prod_{p:\text{prime}} \det(1 - p^{-s} M_p(f) \otimes M_p(g) \otimes M_p(h))^{-1}.$$

For Siegel modular forms $f_1, \cdots, f_m$ and a field K, we denote by $K(f_1, \cdots, f_m)$ the field generated by all the Fourier coefficients of $f_1, \cdots, f_m$ over K. If f and g are C^∞-modular forms (of degree one), we put

$$\langle f, g \rangle_k = \int_{\Gamma_1 \backslash H_1} f(x+iy) \overline{g(x+iy)} y^{k-2} dx dy$$

provided that it converges absolutely. For even integers $r \geq 0$, $k > 4$ and $f \in S_{k+r}(\Gamma_1)$, we denote by $[f]_r$ the Klingen type Eisenstein series attached to f and of type $\det^k \otimes \mathrm{Sym}^r\, \mathrm{St}$, which is a Siegel modular form of degree two. (Precise definition is given later.)

Theorem A. *Let k, l and m be even integers satisfying $k \geq l \geq m$ and $l + m - k > 4$. Let $f \in S_k(\Gamma_1)$, $g \in S_l(\Gamma_1)$ and $h \in S_m(\Gamma_1)$ be normalized Hecke eigenforms. Put*

$$\tilde{L}(s; f, g, h) = \Gamma_{\boldsymbol{C}}(s) \Gamma_{\boldsymbol{C}}(s-k+1) \Gamma_{\boldsymbol{C}}(s-l+1) \Gamma_{\boldsymbol{C}}(s-m+1) L(s; f, g, h)$$

where $\Gamma_{\boldsymbol{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. Then $\tilde{L}(s; f, g, h)$ meromorphically extends to the whole s-plane and satisfies the functional equation

$$\tilde{L}(s; f, g, h) = -\tilde{L}(k+l+m-2-s; f, g, h).$$

Moreover, we have

$$(1) \qquad \pi^{5+k-3l-3m} L(l+m-2; f, g, h) / (\langle f, f \rangle_k \langle g, g \rangle_l \langle h, h \rangle_m) \in \boldsymbol{Q}([f]_{2k-l-m}, f, g, h)$$

and, if $L((k+l+m)/2-1; f, g, h)$ is finite,

$$L\left(\frac{k+l+m}{2} - 1; f, g, h\right) = 0.$$

Corollary. *Let $f \in S_k(\Gamma_1)$ be a normalized Hecke eigenform and $L_3(s, f)$ its third L-function. Put*

$$\tilde{L}_3(s, f) = \Gamma_{\boldsymbol{C}}(s) \Gamma_{\boldsymbol{C}}(s-k+1) L_3(s, f).$$

Then $\tilde{L}_3(s, f)$ satisfies the functional equation