110. On a Closed Range Property of a Linear Differential Operator

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The purpose of this note is to prove the closed range property of a linear differential operator P acting on the space $\mathcal{A}(K)$ of real analytic functions on a compact subset K of $\boldsymbol{R}^n$ under the condition which we call the uniform P-convexity of K. Kiro [6] has recently claimed a similar result, but his reasoning contains serious gaps. In connection with this fact, the first named author (T. K.) wants to replace the condition (1.2) in his announcement paper [4] by the condition (1) below. See Kawai [5] for details.

To state our result, let us first prepare some notations. Let $P(x, D_x)$ be a linear differential operator with (not necessarily real-valued) real analytic coefficients defined on an open neighborhood U of K. Let $p_m(x, \xi)$ denote the principal symbol of $P(x, D_x)$ and suppose that it has a form $q(x, \xi)^l$ for a positive integer l, where $q(x, \xi)$ is a real analytic function in (x, ξ) that is a homogeneous polynomial of ξ of degree $r(=m/l)$. Then the set K is said to be uniformly P-convex if $K=\{x \in U;\ \psi(x) \leqq 0\}$ holds for a real-valued real analytic function $\psi(x)$ which is defined on U satisfying the following condition (1) with some strictly positive constants A_0 and C:

(1) Setting $z=x+\sqrt{-1}y$ and $\zeta=\dfrac{1}{2}\operatorname{grad}\psi(x)-\sqrt{-1}Ay$, we find

$$\sum_{1\leqq j,k\leqq n}\frac{1}{2}\frac{\partial^2\psi(x)}{\partial x_j\partial x_k}q^{(j)}(z,\zeta)\overline{q^{(k)}(z,\zeta)}+\operatorname{Re}\left(\sum_{j=1}^n q_{(j)}(z,\zeta)\overline{q^{(j)}(z,\zeta)}\right)$$
$$-\sum_{j=1}^n |q_{(j)}(z,\zeta)|^2/A\geqq C(1+A|y|)^{2(r-1)}$$

for $A>A_0$, on the condition that $q(z,\zeta)=0$ and $A\psi(x)+A^2|y|^2=1$.

Here, and in what follows, $q^{(j)}(z,\zeta)$ (resp., $q_{(j)}(z,\zeta)$) denotes $\partial q/\partial\zeta_j$ (resp., $\partial q/\partial z_j$).

Remark. It seems to be interesting that the uniform P-convexity is quite akin to the strong P-convexity which Hörmander [1] used to obtain a priori estimates of solutions.

Now, our result is the following

Theorem. *Let K be a compact subset of $\boldsymbol{R}^n$ and let $P(x, D_x)$ be a linear differential operator defined on an open neighborhood U of K. Suppose that K is uniformly P-convex. Then $P\mathcal{A}(K)$ is a closed subspace of $\mathcal{A}(K)$.*

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