107. A Characterization of Chebyshev Spaces

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§1. **Introduction.** Let M be a finite dimensional linear subspace of $C[a, b]$, the space of real valued continuous functions defined on a finite closed interval $[a, b]$. Then, for a function $f \in C[a, b]$, we are concerned with the approximation problem:

$$\text{find } \tilde{f} \in M \text{ to minimize } \|f-\tilde{f}\|,$$

where $\|\cdot\|$ denotes the uniform norm. The function $\tilde{f} \in M$ is said to be a best approximation to f from M if $\tilde{f}$ is a solution to the above problem. For an n-dimensional subspace M, we put the following two subsets of $C[a, b]$: $U_M=\{f \mid f$ possesses a unique best approximation$\}$ and $A_M=\{g \mid$ the error function $e=g-\tilde{g}$ has an alternating set of $(n+1)$ points in $[a, b]$ for any best approximation $\tilde{g}$ to g; i.e., there exist $(n+1)$ distinct points $a \leqq x_1 < \cdots < x_{n+1} \leqq b$ such that $|e(x_i)|=\|e\|$, $i=1, 2, \cdots, n+1$ and $e(x_i) \cdot e(x_{i+1}) \leqq 0$, $i=1, \cdots, n\}$.

As is well known, if M is a Chebyshev space (respectively weak Chebyshev space), that is, every nonzero function in M has no more than $n-1$ zeros (respectively changes of sign) on $[a, b]$, then they are of great use in this problem. Hence various properties and characterizations of these spaces have been obtained. Young [5] showed that if M is a Chebyshev space then U_M is equal to $C[a, b]$. Further, by the result of Haar [1], a necessary and sufficient condition that M is a Chebyshev space is that U_M coincides with $C[a, b]$.

As a characterization of a weak Chebyshev space, Jones and Karlovitz [2] proved that M is a weak Chebyshev space if and only if U_M is included in A_M. In this paper, as the above result, we shall give a characterization of a Chebyshev space M by using an inclusion relation between U_M and A_M.

§2. **Definitions and lemmas.** In this section, we prepare several lemmas necessary for the proof of the main theorem. First we begin with some definitions.

Definition 1. For a function $f \in C[a, b]$, two zeros x_1, x_2 of f are said to be *separated* if there is an x_0, $x_1 < x_0 < x_2$, such that $f(x_0) \neq 0$.

For an n-dimensional subspace M of $C[a, b]$, we define the followings.

Definition 2. (i) We call a point $x_0 \in [a, b]$ *vanishing* with respect to M if $g(x_0)=0$ for any $g \in M$. In case that no confusion arises, the term "with respect to M" will be omitted.

(ii) M is called *vanishing* if there exists at least one vanishing point in $[a, b]$. Otherwise, it is called *nonvanishing*.