104. On the Derived Categories of Mixed Hodge Modules

By Morihiko SAITO
Research Institute for Mathematical Sciences, Kyoto University

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Let X be a nonsingular (separated) algebraic variety over $\boldsymbol{C}$, and $MHM(X, \boldsymbol{Q})$ the abelian category of mixed Hodge Modules [5]. For simplicity, $MHM(X, \boldsymbol{Q})$ will be denoted by $MHM(X)$. Let $D^b MHM(X)$ be the derived category of bounded complexes of $MHM(X)$. Then $D^b MHM(X)$ are stable by the functors: $f_*, f_!, f^*, f^!, \psi_g, \varphi_{g,1}, \xi_g$ (cf. 1.1), $\boldsymbol{D}$ and $\boxtimes$. I would like to thank Prof. Kashiwara for useful and stimulating discussions.

§ 1. Vanishing cycle functors.

1.1. Let g be a function on X. By definition (cf. [5]) we have the exact functors

$$\psi_g : MHM(X) \longrightarrow MHM(X), \qquad \varphi_{g,1} : MHM(X) \longrightarrow MHM(X).$$

We define a functor $\xi_g : MHM(X) \to MHM(X)$ as follows:

Let $j_g : \{g \neq t\} \to X \times \boldsymbol{C}$ be the open immersion, and $p : X \times \boldsymbol{C} \to X$ the projection, where t is the coordinate of $\boldsymbol{C}$. Then we define

$$\xi_g = \psi_{t,1} j_{g!} j_g^* p^*[1].$$

Note that the functors $j_{g!}$ and $p^*[1]$ exist by definition [5].

1.2. Proposition. *We have the functorial exact sequences:*

$$0 \longrightarrow \psi_{g,1}\mathcal{M} \longrightarrow \xi_g \mathcal{M} \longrightarrow \mathcal{M} \longrightarrow 0$$
$$0 \longrightarrow j_! j^* \mathcal{M} \longrightarrow \xi_g \mathcal{M} \longrightarrow \varphi_{g,1}\mathcal{M} \longrightarrow 0$$

for $\mathcal{M} \in MHM(X)$, *where* $j : X \setminus g^{-1}(0) \to X$.

1.3. Remark. Beilinson's functor Ξ_g used in [1] should correspond to $\xi_g j_*$.

1.4. Corollary. *Let Z be a closed (reduced) subvariety of X, and $MHM_Z(X)$ (resp. $D^b_Z MHM(X)$) the full subcategory of $MHM(X)$ (resp. $D^b MHM(X)$) of the objects with supports (resp. cohomological supports) in Z. Then*

$$D^b MHM_Z(X) \longrightarrow D^b_Z MHM(X)$$

is an equivalence of categories.

This follows from 1.2. by the same argument as in [1], because the adjunction $\mathrm{Hom}(j^*\mathcal{M}^\cdot, \mathcal{N}^\cdot) \simeq \mathrm{Hom}(\mathcal{M}^\cdot, j_*\mathcal{N}^\cdot)$ for an affine open immersion j follows from the existence of the natural morphism $\mathcal{M}^\cdot \to j_* j^* \mathcal{M}^\cdot$.

§ 2. Duals.

2.1. Proposition. *$MHM(X)$ (hence $D^b MHM(X)$) is stable by the dual functor $\boldsymbol{D}$.*

This follows from the compatibility of the algebraic and topological dualities with respect to the functors ψ, φ_1.