String Theories and Prime Number Theories

By Nobushige KUROKAWA
Department of Mathematics, Tokyo Institute of Technology

String theories (cf. Green-Gross [5]) seem to suggest much to prime number theories. We supplement the previous report [7] from the viewpoint of string theories. In § 1 we notice an application of string theories, and we remark the "multiple string case" in § 2. The contents of this report were presented in a symposium entitled "Superstring Theories" at the University of Tokyo in September 1986. Some details will appear elsewhere.

§ 1. An application of string theories. Let $M$ be a compact Riemann surface of genus 2, $\tau_M$ the period matrix belonging to the Siegel upper half space of genus 2, and $Z_M(s)$ the original Selberg zeta function. Let $\tau_{10}$ be the Siegel cusp form of genus 2 and weight 10 uniquely determined up to constant: $\tau_{10} = c \prod_{m \text{even}} \sigma_m^r$. Then:

**Theorem 1.** $Z_M'(1)Z_M(2)^{-1} = C |\tau_{10}(\tau_M)^\frac{3}{2} (\det \text{Im} \tau_M)^{10}$ up to an absolute constant $C$ independent of $M$.

This follows from the recent progress of string theories. First, D’Hoker-Phong [4] (cf. [1]) showed that:

$Z_M'(1)Z_M(2)^{-1} = C_1 (\det J_M)^{13} (\det J_M^{10})^{-1}$,

where $J_M$ is the usual Laplacian on $M$ and $J_M^{10}$ is a Laplacian acting on certain tensors. Secondly, Belavin-Knizhnik [3] (cf. [2], [6], [10]) showed that:

$|\tau_{10}(\tau_M)|^\frac{3}{2} (\det \text{Im} \tau_M)^{10} = C_2 (\det J_M)^{13} (\det J_M^{10})^{10}$. These two results are proved by quite different methods, and Theorem 1 seems to be astonishing.

We notice another suggestion from string theories, which is conjecturally schematized as follows:

The left tree indicates the unification of the four forces: weak, electromagnetic, strong, and gravitational. The right tree indicates the unification of the four zeta functions: Artin, Hecke-Langlands, Hasse-Weil, and