84. *Some New Two-step Integration Methods*

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1. Introduction. The purpose of this is to present some new two-step methods, which deal with the following initial value problem:

(1.1) $$y'=f(x, y), \quad y(x_0)=y_0.$$

Of all computational methods for (1.1), Runge-Kutta (abbr., R-K) are most popular. R-K methods retain the advantage of one-step methods, but need some functional evaluations for each step. We shall look for other methods to decrease the functional evaluations in R-K methods. Such methods have been discussed by Byrne, Lambert [1] and many others. We have seen in [1] that two-step R-K methods have order $p(r)=r+1$ $(r=2, 3, 4)$, and that R-K methods [2], [3] have order $p(r)=r$ $(r=1, 2, 3, 4)$, $p(5)=4$, $p(6)=5$, $p(r)=6$ $(r=7, 8)$, $p(r)=7$ $(r=9, 10)$, $p(11)=8$, where $p(r)$ denotes the highest order that can be attained by an r-stage. Thus two-step R-K methods attain higher order than R-K methods for the same stage. However, in actual computation, two-step R-K methods would not yield as good numerical results as R-K methods for the same order, and some people seem to have the opinion that two-step R-K methods may not be useful for actual computations, but some useful two-step methods are still required in many fields. We now propose the following two-step R-K methods which improve the defect of the usual two-step R-K methods:

(1.2)
$$\begin{aligned}
&y_{n+1}=V_1^{(1)}y_{n-1}+V_2^{(1)}y_n+h\Phi^{(1)}(x_{n-1}, x_n, y_{n-1}, y_{n-1+\theta_1}, y_n, y_{n+\theta_2}; h),\\
&y_{n+1+\theta}=V_1^{(2)}y_{n-1}+V_2^{(2)}y_n+h\Phi^{(2)}(x_{n-1}, x_n, y_{n-1}, y_{n-1+\theta_1}, y_n, y_{n+\theta_2}; h),\\
&\Phi^{(j)}(x_{n-1}, x_n, y_{n-1}, y_{n-1+\theta_1}, y_n, y_{n+\theta_2})=\textstyle\sum_{i=1}^{r}(W_i^{(j)}k_i(x_{n-1})+S_i^{(j)}k_i(x_n))\\
&\qquad\qquad (0\le\theta=\theta_1,\ \theta_2\le 1), \quad (j=1, 2),\\
&k_1(x_{n-j})=f(x_{n-j}, y_{n-j}) \quad (j=0, 1),\\
&k_i(x_{n-1})=f(x_{n-1}+a_ih,\ y_{n-1}+b_iy_{n-1+\theta_1}+h\textstyle\sum_{j=1}^{r-1}b_{ij}k_j(x_{n-1})),\\
&k_i(x_n)=f(x_n+c_ih,\ y_n+d_iy_{n+\theta_2}+h\textstyle\sum_{j=1}^{r-1}d_{ij}k_j(x_n)),\\
&a_i=b_i+\textstyle\sum_{j=1}^{r-1}b_{ij}, \quad c_i=d_i+\sum_{j=1}^{r-1}d_{ij} \quad (0<a_i,\ c_i\le 1).
\end{aligned}$$

In our methods, we have $p(2)=5$. In using our method, we assume that we have already computed the value of $y(x_0+\theta h)$, $y(x_0+h)$ and $y(x_0+(1+\theta)h)$ by some other means, where $y(x)$ denotes the analytical solutions of (1.1). We first calculate the value of y_1 and $y_{1+\theta_1}$ by some means of (1.2), and next proceed to the calculation of y_2 and $y_{2+\theta_2}$. To demonstrate

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