83. *Boundedness of Closed Linear Operator T satisfying* $R(T)\subset D(T)$

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1. Let T be a densely defined closed linear operator in a Banach space E satisfying that $R(T)\subset D(T)$. We prove that if T satisfies one of the following conditions:

(1) $\|T^2x\|\|x\|\geqq\|Tx\|^2$ for every $x\in D(T)$, or

(2) T has the non-empty resolvent set,

then it follows that T is bounded.

Let $T:E\to H$ be a densely defined closed linear operator and $T^*:H\to E'$ be the adjoint operator, where H is a Hilbert space. It is shown that if $R(T)\subset D(T^*)$, then T is bounded.

2. Let E be a Banach space and $T:E\to E$ be a densely defined closed linear operator with the domain $D(T)$ and the range $R(T)$. The following problem was posed by Ôta [3].

Problem. *Suppose that $R(T)\subset D(T)$, then is T bounded?*

In general, the answer is negative as shown by Ôta [3]. Ôta [3] proved that if T is dissipative, then the answer is positive. In this note we investigate other conditions which imply the positive answers for this problem.

After Furuta [1], we say the linear operator $T:E\to E$ *paranormal* if $R(T)\subset D(T)$ and if it holds that $\|T^2x\|\|x\|\geqq\|Tx\|^2$ for every $x\in D(T)$.

Theorem 1. *Let T be a densely defined closed paranormal operator in a Banach space E. Then T is bounded.*

Proof. Since T is closed, $(D(T),|\ |_T)$ is a Banach space, where $|x|_T=\|x\|+\|Tx\|$, $x\in D(T)$. By $R(T)\subset D(T)$, the operator $T^2:(D(T),|\ |_T)\to E$ is well defined. By the closedness of T, it follows that T^2 is also closed on $(D(T),|\ |_T)$, hence bounded. Thus there exists $C>0$ such that $\|T^2x\|\leqq C(\|x\|+\|Tx\|)$ for every $x\in D(T)$. By the paranormality, we have for every $x\in D(T)$ with $\|x\|=1$, $\|Tx\|^2\leqq\|T^2x\|\leqq C(1+\|Tx\|)$. That is, $\|Tx\|^2-C\|Tx\|-C\leqq 0$. This implies that

$$\|Tx\|\leqq\frac{C+\sqrt{C^2+4C}}{2}<+\infty,$$

which implies the assertion.

Let T be a linear operator in a Banach space E. The *resolvent set* $\rho(T)$ of T is the set of all complex numbers λ such that the range $R(\lambda I-T)$ is dense in E and that $\lambda I-T$ has the continuous inverse $(\lambda I-T)^{-1}$ on $D((\lambda I-T)^{-1})=R(\lambda I-T)$, see Yosida [4], Ch. VIII. It is well known that if T is bounded, then $\rho(T)\neq\emptyset$. The converse is valid for a densely defined