77. On Conjugacy Classes of the Pro-l braid Group of Degree 2[†)]

By Masanobu KANEKO
Department of Mathematics, Faculty of Science, University of Tokyo

(Communicated by Shokichi IYANAGA, M. J. A., Sept. 12, 1986)

0. Introduction. In [2], Y. Ihara studied the "pro-l braid group" of degree 2 which is a certain big subgroup $\Phi \subset \mathrm{Out}\,\mathfrak{F}$ of the outer automorphism group of the free pro-l group $\mathfrak{F}$ of rank 2. There is a canonical representation $\varphi_Q : G_Q \to \Phi$ of the absolute Galois group $G_Q = \mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ which is unramified outside l, and for each prime $p \neq l$, the Frobenius of p determines a conjugacy class C_p of Φ which is contained in the subset $\Phi_p \subset \Phi$ formed of all elements of "norm" p (loc. cit. Ch. I). In this note, we shall prove that Φ_p contains *infinitely* many Φ-conjugacy classes, at least if p generates $\boldsymbol{Z}_l^\times$ topologically. It is an open question whether one can *distinguish* the Frobenius conjugacy class from other norm-p-conjugacy classes.

1. The result. Let l be a rational prime. We denote by $\boldsymbol{Z}_l$, $\boldsymbol{Z}_l^\times$ and $\boldsymbol{Q}_l$, respectively, the ring of l-adic integers, the group of l-adic units and the field of l-adic numbers. As in [2], let $\mathfrak{F} = \mathfrak{F}^{(2)}$ be the free pro-l group of rank 2 generated by x, y, z, $xyz = 1$, $\Phi = \mathrm{Brd}^{(2)}(\mathfrak{F}; x, y, z)$ be the pro-l braid group of degree 2, $\mathrm{Nr}(\sigma) \in \boldsymbol{Z}_l^\times$ be the norm of $\sigma \in \Phi$, and for $\alpha \in \boldsymbol{Z}_l^\times$, Φ_α be the "norm-α-part", i.e., $\Phi_\alpha = \{\sigma \in \Phi \mid \mathrm{Nr}(\sigma) = \alpha\}$.

Theorem. *If $\alpha \in \boldsymbol{Z}_l^\times$ generates $\boldsymbol{Z}_l^\times$, then the set Φ_α contains infinitely many Φ-conjugacy classes.*

Remarks. 1) In [2], it is proved under the same assumption, that Φ_α contains at least two Φ-conjugacy classes. (Corollary of Proposition 8, Ch. I.)

2) In [1], M. Asada and the author studied the "pro-l mapping class group" and obtained a result similar to 1).

2. Proof. Our method of proof is to consider the projection of Φ to the group $\Psi = \mathrm{Brd}^{(2)}(\mathfrak{F}/\mathfrak{F}''; x, y, z)$, where $\mathfrak{F}'' = [\mathfrak{F}', \mathfrak{F}']$, $\mathfrak{F}' = [\mathfrak{F}, \mathfrak{F}]$ and we use the same symbols x, y, z for their classes mod $\mathfrak{F}''$. By Theorem 3 in [2] Ch. II, the group Ψ is explicitly realized as follows. Define the group Θ by

$$\Theta = \{(\alpha, F) \mid \alpha \in \boldsymbol{Z}_l^\times,\ F \in \mathcal{A}^\times,\ F + uvw\mathcal{A} = \theta_\alpha\}$$

with the composition law $(\alpha, F)(\beta, G) = (\alpha\beta, F \cdot G^{j_\alpha})$, where

$$\mathcal{A} = \boldsymbol{Z}_l[[u, v, w]]/((1+u)(1+v)(1+w)-1) \simeq \boldsymbol{Z}_l[[u, v]],$$

†) This is a part of the master's thesis of the author at the University of Tokyo (1985). He wishes to express his sincere gratitude to Professor Y. Ihara for his advice and encouragement.