76. On Cusp Forms of Octahedral Type

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1. In [1], Chinburg formulated Stark conjecture "over $\boldsymbol{Z}$". He proved for one dimensional case and gave several examples for two dimensional representations of tetrahedral type. The purpose of this note is to give an example of octahedral type.

First recall some notations of [1]. Let K be a finite Galois extension of $\boldsymbol{Q}$ with Galois group G. Let S_∞ be the set of infinite places of K and let $\| \ \|_v$ be the normalized absolute value for $v \in S_\infty$. Let $\boldsymbol{V}=(V, S)$ denote the pair of an irreducible complex representation V of G and a finite set S of places of $\boldsymbol{Q}$ containing ∞. Define $L(s, \boldsymbol{V})=\prod_{p \notin S} L_p(s)$ where $L_p(s)$ is a usual Euler p-factor. Let χ_V be the character of V and put $\mathrm{pr}_{\boldsymbol{V}}=\sum_{g \in G} \chi_V(g) g$. Let $L_{\boldsymbol{V}}(s)=L(s, \boldsymbol{V})\,\mathrm{pr}_{\boldsymbol{V}}$ and $L'_{\boldsymbol{V}}(s)=L'(s, \boldsymbol{V})\,\mathrm{pr}_{\boldsymbol{V}}$. We only consider representations V such that $L(s, \boldsymbol{V})$ has a zero of order 1 at $s=0$.

Let $d=\sum d_{\boldsymbol{V}} \boldsymbol{V}$ be a finite linear combination of $\boldsymbol{V}$ of dimension n. We define $L(s, d)=\sum d_{\boldsymbol{V}} L(s, \boldsymbol{V})$, $L'(s, d)=\sum d_{\boldsymbol{V}} L'(s, \boldsymbol{V})$, $L_d(s)=\sum d_{\boldsymbol{V}} L_{\boldsymbol{V}}(s)$ and $L'_d(s)=\sum d_{\boldsymbol{V}} L'_{\boldsymbol{V}}(s)$. The element $\tau \in \mathrm{Aut}\,(\boldsymbol{C}/\boldsymbol{Q})$ acts on d by $d^\tau=\sum d_{\boldsymbol{V}}^\tau \boldsymbol{V}^\tau$ where $\boldsymbol{V}^\tau=(V^\tau, S)$ for $\boldsymbol{V}=(V, S)$. We define the additive group $\boldsymbol{D}(n)$ by the set of all d such that $d^\tau=d$ for any $\tau \in \mathrm{Aut}\,(\boldsymbol{C}/\boldsymbol{Q})$ and such that $L(s, d)$ is a Dirichlet series with integral coefficients.

Stark Conjecture. Suppose that $n=1$ or 2, and $d \in \boldsymbol{D}(n)$. Then $\exp\,(L'(0, d))=e(d)$ is a real unit in K, and $L'_d(0) v_0=\sum_{v \in S_\infty} \log \| e(d) \|_v\, v$ where v_0 is a fixed embedding of K into $\boldsymbol{C}$. If $n=2$, the real conjugates of $e(d)$ are positive.

2. From now on, we consider the space of cusp forms of weight 1 on $\Gamma_0(283)$ with the character $(-283/*)$. This space has one primitive form of S_3 type and two primitive forms of S_4 type (cf. Serre [2]). Let h be of S_3 type and let f be one of the forms of S_4 type. Then the other form of S_4 type is f^ρ where ρ is a complex conjugation. The Fourier coefficients of h and f are listed in the table below. Numerical computation shows that

$$L'(0, f)=2.4681497509\cdots+0.2223673138\cdots i,$$
$$L'(0, h)=2.802684\cdots.$$

Let V_f and W be Galois representations attached to f and h respectively. We denote by φ the projection $GL_2(\boldsymbol{C}) \to PGL_2(\boldsymbol{C})$, and put $\tilde{V}_f=\varphi \cdot V_f$ and $\tilde{W}=\varphi \cdot W$. Let K and L be the fields corresponding to the kernels of V_f and $\tilde{V}_f$ respectively. Then