74. Group Rings whose Augmentation Ideals are Residually Lie Solvable

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1. Introduction. Let R be a commutative ring with identity and G be a group. We denote the augmentation ideal of the group ring RG by $\Delta_R(G)$. There are many problems and results relating to $\Delta_R(G)$ (cf. [6]). In particular, it is an interesting problem to characterize the group rings whose augmentation ideals satisfy some conditions. In this paper, we treat the Lie property. We recall some definitions. Let S be a ring and I be a two sided ideal of S. Then $I^{\langle n\rangle}$ and $I^{(n)}$ are the ideals of S defined inductively as follows, respectively.

$$I^{\langle 1\rangle}=I, \qquad I^{\langle n+1\rangle}=[I^{\langle n\rangle}, I^{\langle n\rangle}]S$$
$$I^{(1)}=I, \qquad I^{(n+1)}=[I, I^{(n)}]S,$$

where $[M, N]$ is the additive subgroup of S generated by the elements of the form $[m, n]=mn-nm$ with $m \in M$ and $n \in N$. We say that I is *Lie solvable* (resp. *Lie nilpotent*) if $I^{\langle n\rangle}=0$ for some n (resp. $I^{(n)}=0$ for some n). And I is called *residually Lie solvable* (resp. *residually Lie nilpotent*) if $\bigcap_n I^{\langle n\rangle}=0$ (resp. $\bigcap_n I^{(n)}=0$).

Parmenter-Passi-Sehgal [5] characterizes those groups G such that $\Delta_R(G)$ is Lie nilpotent. The condition under which $\Delta_k(G)$ is residually Lie nilpotent when k is a field is also known (cf. [6]). Further, Musson-Weiss [4] gave the characterization of the groups G such that $\Delta_Z(G)$ is residually Lie nilpotent. In [7], the groups G such that RG is Lie solvable are characterized (Lie solvability in our sense is called "strong" Lie solvability in that book). On the other hand, we have $\Delta_R^{\langle n\rangle}(G)=RG^{\langle n\rangle}$ and $\Delta_R^{(n)}(G)=RG^{(n)}$ because $[x, y]=[x-\varepsilon(x)\cdot 1, y-\varepsilon(y)\cdot 1]$ where $x, y \in RG$ and ε is the augmentation map. Thus those groups G such that $\Delta_R(G)$ is Lie solvable are already characterized. Now the aim of this paper is to show the following

Theorem. *Let G be a finite group. Then $\bigcap_n \Delta_Z^{\langle n\rangle}(G)=0$ if and only if G' is a p-group for some prime p, where G' is the commutator subgroup of G.*

2. Preliminaries. The following is the key lemma to prove our theorem.

Lemma. *Let R be a commutative ring with identity and G be a finite group. Let K, L be the subgroups of G such that $K \leqq L \leqq N_G(K)$ and put $N=(K, L)=\langle k^{-1}l^{-1}kl \mid k \in K, l \in L\rangle$. Then for any $x \in N$ and $n \geqq 2$, we have*

$$(*) \qquad |N|^{2^{n-1}-2}(x-1) \in \Delta_R^{\langle n\rangle}(G).$$

Proof. We use the induction on n. Since