68. 2-nd Microlocalization and Conical Refraction

By Nobuyuki Tose
Department of Mathematics, Faculty of Science, Ehime University
(Communicated by Kōsaku Yosida, M. J. A., Sept. 12, 1986)

§ 1. Introduction. The phenomenon of conical refraction has long been observed by physicists. It is attributed to the non-uniformity of multiplicities to Maxwell equation in the crystal and studied in the framework of Microlocal Analysis by Melrose-Uhlmann [8] and P. Laubin [5], [6].

We employ the theory of 2-microlocalization developed by M. Kashiwara and Y. Laurent (see [2], [4]) and gain a new insight about the phenomenon.

Explicitly, let $P$ be a microdifferential operator defined in a neighborhood of $\rho_0 \in \sqrt{-1}\tilde{T}^*R^n$, which satisfies the following conditions.

1. $P$ has a real principal symbol $p$.

Let $\Sigma_1 = \{ \rho \in \sqrt{-1}\tilde{T}^*R^n ; p(\rho) = 0 \}$ and $\Sigma_2 = \{ \rho \in \Sigma_1 ; dp(\rho) = 0 \}$.

2. $\Sigma_2$ is a regular involutory submanifold in $\sqrt{-1}\tilde{T}^*R^n$ through $\rho_0$ of codimension $d \geq 3$.

3. $\text{Hess } p(\rho)$ has rank $d$ with positivity 1 if $\rho \in \Sigma_2$.

Moreover we assume

4. $P$ has regular singularities along $\Sigma_2^C$ in the sense of Kashiwara-Oshima [3], where $\Sigma_2^C$ denotes a complexification of $\Sigma_2$ in $T^*C^n$.

Our main interest is the propagation of singularities on $\Sigma_2$ for the equation $Pu = 0$, which can be transformed by a quantized contact transformation into

\begin{equation}
Pu = \left( D^2 - \sum_{i,j=2}^d A^{ij}(x, D)D_iD_j + \text{(lower)} \right) u = 0.
\end{equation}

defined in a neighborhood of $\rho_1 = (0, \sqrt{-1}dx_2)$. Here $A^{ij}$ are of order 0 with $(\sigma(A^{ij}))$ positive definite. We remark that in this case $\Sigma_2 = \{(x, \sqrt{-1}\xi) ; \xi_1 = \cdots = \xi_d = 0 \}$ and that $P_0$ has regular singularities along $\Sigma_2^C$.

We study (5) 2-microlocally along $\Sigma_2$. After transforming (5) by a quantized homogeneous bicanonical transformation, which is wider than quantized contact transformations, we give the canonical form of (5) as $D_iu = 0$. Then we can easily obtain a theorem about the propagation of 2-microlocal singularities.

§ 2. Notation. Let $X$ be a complex manifold and $\Lambda$ be a regular involutory submanifold of $T^*X$. $\Lambda$ is embedded naturally into $\Lambda \times \Lambda$. $\tilde{\Lambda}$ denotes the union of all bicharacteristics of $\Lambda \times \Lambda$ that pass through $\Lambda$. $\mathcal{E}_{\tilde{\Lambda}}^\infty$ is the sheaf on $T^*_\Lambda \tilde{\Lambda}$ of 2-microdifferential operators constructed by Y. Laurent [4].

Let $M$ be a real analytic manifold whose complexification is $X$. $\Sigma$ denotes a regular involutory submanifold of $T^*_\Sigma X$, whose complexification