64. Tori whose Covering Spaces have Convex Distance Functions

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0. Introduction. E. Hopf ([4]) proved that Riemannian tori T^2 without conjugate points are flat. The theorem has no analogue in the G-space theory of Busemann ([1]). Namely, H. Busemann ([1], p. 223) has proved that there are metrizations of the torus without conjugate points for which the universal covering space is not Minkowskian. Recently, N. Innami ([5]) proved that Riemannian tori T^n, $n\geq 2$, are flat if there is a point which cannot be a focal point of any geodesic (as a 1-dimensional submanifold). In the present note we shall show that this has an analogue in G-surfaces. The significance of G-spaces can be seen in [1], Section 15.

Let M be a G-space and let $f: M\to \boldsymbol{R}$ be a function. We say that f is *convex* on M if $f\circ\alpha$ is a one-variable convex function for any geodesic $\alpha:(-\infty,\infty)\to M$.

Theorem. *Let N be a G-space which is homeomorphic to the torus T^2 and let M be its universal covering G-space. If M has a point o such that the distance function from o is convex on M, then M is Minkowskian.*

If a compact Riemannian manifold has a non-focal point, then the manifold has no focal points ([6]). And a simply connected Riemannian manifold has no focal points if and only if all distance functions are convex. However, this is not true in the G-space theory. Therefore, we use convex distance functions instead of non-focality properties. We shall show in Section 1 that M is straight, i.e., all geodesics are minimizing in M, and that the distance function from any point is convex on M. Then, combined with the two results, (33.1) in p. 215 and (25.6) in p. 157, [1], these conclude the theorem.

1. Proof. We first prove that o is a pole in M, i.e., all geodesics emanating from o is minimizing. Let $\gamma:[0,\infty)\to M$ be a geodesic with $\gamma(0)=o$. Put $f(t)=d(o,\gamma(t))$ for any $t\in[0,\infty)$. Since f is convex and $f(0)=0$,

$$f(t)\geq f'_+(0)t=t$$

for all $t\geq 0$, where $f'_+(0)$ is the right derivative of f at 0 and, hence, $f'_+(0)=1$. This implies that

$$d(o,\gamma(t))=f(t)=t$$

for all $t\geq 0$, because generally $f(t)\leq t$.

Let D be the group of isometries of M such that $M/D=N$. Then, it follows from Proposition 4.1 in [5] that the displacement functions of all