55. *On Some Integral Invariants on Complex Manifolds.* I

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This note is a continuation of our preceding works (cf. Bando [1], Mabuchi [8]) and we here explain how Futaki invariants (cf. Futaki [5], Futaki and Morita [6]) are generalized and reinterpreted from our view-points. Most of the proofs down below are very sketchy and a complete account including the present results will be given in a separate paper [2].

(I) Fix an arbitrary compact complex r-dimensional connected manifold X. Let $G:=\mathrm{Aut}(X)$ be the group of all holomorphic automorphisms of X and $G^0:=\mathrm{Aut}^0(X)$ be its identity component. We denote by $\mathcal{V}_X$ the set of all volume forms Ω on X such that $\int_X \Omega=1$. Now, to each pair $(\Omega', \Omega'')\in\mathcal{V}_X\times\mathcal{V}_X$, we associate the real number $N_X(\Omega', \Omega'')\in\boldsymbol{R}$ by

$$N_X(\Omega', \Omega''):=\int_a^b dt\int_X \{(\sqrt{-1}/2\pi)\bar\partial\partial\log(\Omega_t)\}^r(\partial\Omega_t/\partial t)/\Omega_t,$$

where $\{\Omega_t\,|\,a\leqq t\leqq b\}$ is an arbitrary piecewise smooth path in $\mathcal{V}_X$ such that $\Omega_a=\Omega'$ and $\Omega_b=\Omega''$. Then by a result of Donaldson [4; Proposition 6] applied to the anti-canonical bundle K_X^{-1} of X, the number $N_X(\Omega', \Omega'')$ above is independent of the choice of the path $\{\Omega_t\,|\,a\leqq t\leqq b\}$ and therefore well-defined. Furthermore, N_X is G-invariant, i.e.,

$$N_X(g^*\Omega', g^*\Omega'')=N_X(\Omega', \Omega'') \qquad \text{for all } g\in G \text{ and all } \Omega', \Omega''\in\mathcal{V}_X,$$

and satisfies the 1-cocycle condition, i.e.,

(i) $N_X(\Omega', \Omega'')+N_X(\Omega'', \Omega')=0$ and

(ii) $N_X(\Omega, \Omega')+N_X(\Omega', \Omega'')+N_X(\Omega'', \Omega)=0$,

for all $\Omega, \Omega', \Omega''\in\mathcal{V}_X$. We now fix an arbitrary element Ω_0 of $\mathcal{V}_X$, and define a functional $\nu_X:\mathcal{V}_X\to\boldsymbol{R}$ by

$$\nu_X(\Omega):=N_X(\Omega_0, \Omega), \qquad \Omega\in\mathcal{V}_X.$$

We moreover set

$$n_X(g):=\exp(\nu_X(g^*\Omega_0)), \qquad g\in G.$$

Then the same argument as in [8; § 5] easily allows us to obtain:

Proposition A. (i) *$n_X: G\to\boldsymbol{R}_+$ is a Lie group homomorphism which does not depend on the choice of Ω_0, where $\boldsymbol{R}_+$ denotes the group of positive real numbers. In particular, n_X is trivial on $[G, G]$.*

(ii) *Let $\lambda:=c_1(X)^r[X]$. Then $\Omega\in\mathcal{V}_X$ is a critical point of ν_X if and only if* $\{(\sqrt{-1}/2\pi)\bar\partial\partial\log(\Omega)\}^r=\lambda\Omega$, i.e., $(\sqrt{-1}/2\pi)\bar\partial\partial\log(\Omega)$ *is a (possibly indefinite) Einstein form.*

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