52. *Area Integrals for Normal and Yosida Functions*

By Shinji YAMASHITA
Tokyo Metropolitan University

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1. Introduction. We shall consider necessary and sufficient conditions for a meromorphic function to be normal or Yosida ([1], [6]).

A function f meromorphic in $D=\{|z|<1\}$ is called normal if

$$k(f)=\sup_{z\in D}(1-|z|^2)f^{\#}(z)<\infty,$$

where $f^{\#}=|f'|/(1+|f|^2)$ is the spherical derivative. In terms of the non-Euclidean hyperbolic distance:

$$\sigma(z, w)=\tanh^{-1}|\phi_w(z)|,$$

where

$$\phi_w(z)=(z-w)/(1-\overline{w}z), \qquad z, w\in D,$$

the non-Euclidean open disk of center $a\in D$ and radius $\tanh^{-1}\rho$ $(0<\rho\leqq 1)$ is given by

$$\Delta(a, \rho)=\{|\phi_a(z)|<\rho\}.$$

Theorem 1. *Let f be meromorphic in D. Then the following are mutually equivalent.*

(1) *f is normal.*

(2) *For each $A>0$ there exists $\rho\in(0, 1)$ such that*

$$\text{(1.1)}\qquad \sup_{a\in D}\sup_{z\in\Delta(a,\rho)}\left|\frac{f(z)-f(a)}{1+\overline{f(a)}f(z)}\right|<A.$$

(3) *There exist ρ and λ in $(0, 1)$ such that*

$$\text{(1.2)}\qquad \sup_{\lambda<|a|<1}\iint_{\Delta(a,\rho)}\left|\frac{f(z)-f(a)}{1+\overline{f(a)}f(z)}\right|^2\frac{dxdy}{(1-|z|^2)^2}<\infty.$$

Here, $(f(z)-f(a))/(1+\overline{f(a)}f(z))=1/f(z)$ if $f(a)=\infty$. We note that $(1-|z|^2)^{-2}dxdy$ is the non-Euclidean area element at $z=x+iy\in D$.

A function f meromorphic in $\boldsymbol{C}=\{|z|<\infty\}$ is called Yosida if

$$l(f)=\sup_{z\in\boldsymbol{C}}f^{\#}(z)<\infty.$$

See [2], [3], [4], and [5]. We next consider the Euclidean disks:

$$U(a, \rho)=\{|z-a|<\rho\}, \quad a\in\boldsymbol{C}, \quad \rho>0.$$

Theorem 2. *Let f be meromorphic in $\boldsymbol{C}$. Then the following are mutually equivalent.*

(4) *f is Yosida.*

(5) *For each $A>0$ there exists $\rho\in(0, \infty)$ such that*

$$\text{(1.3)}\qquad \sup_{a\in\boldsymbol{C}}\sup_{z\in U(a,\rho)}\left|\frac{f(z)-f(a)}{1+\overline{f(a)}f(z)}\right|<A.$$

(6) *There exist ρ and λ in $(0, \infty)$ such that*

$$\text{(1.4)}\qquad \sup_{\lambda<|a|<\infty}\iint_{U(a,\rho)}\left|\frac{f(z)-f(a)}{1+\overline{f(a)}f(z)}\right|^2dxdy<\infty.$$