43. A Note on the Mean Value of the Zeta and L-functions. III

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1. In the present note we study the twelfth power moment of $L(1/2+it,\chi)$, χ being primitive character mod q. We restrict ourselves to the case of prime q; this is mostly for the sake of simplicity (cf. Remark below).

We consider

$$I=\int_{T-G}^{T+G}\left|L\left(\frac{1}{2}+it,\chi\right)\right|^2 dt,$$

where

(1) $$q^{1/2}\leqq T,\quad (qT)^{\varepsilon}\leqq G\leqq (qT)^{1/3}l^{-1}\quad (l=\log qT).$$

Using the function E_1 introduced in [4], we have

$$I\ll Gl+\left|\int_{-\infty}^{\infty}E_1(T+t,\chi)tG^{-2}\exp(-(t/G)^2)dt\right|.$$

Then following closely the argument of [4, §2] one may show that for an $N\approx qT$

$$\begin{aligned}I\ll{}& Gl+G^{-1}((qT)^{1/4}+q^{1/2}(qT)^{\varepsilon})l\\&+G\left|\sum_{n\leqq N}a(n,\chi)\int_0^{\infty}(y(y+1))^{-1/2}\cos(T\log(1+1/y))\right.\\&\left.\times\exp\left(-2\pi iny/q-\frac{1}{4}(G\log(1+1/y))^2\right)dy\right|.\end{aligned}$$

To estimate this sum over n, we divide it into two parts according to $qTG^{-2}l^{-2}<n\leqq N$ and $n\leqq qTG^{-2}l^{-2}$. To the integrals in the first sum we apply the second mean value theorem, and find that they are $\ll ql(nG)^{-1}$. Thus by [4, Lemma 5] we see that the first sum is $\ll q^{1/2}G^{-1}l^3$. On the other hand, to the integrals in the second sum we apply the saddle point method, and after overcoming somewhat lengthy computation we find that they are equal to

$$\pi^{1/4}q^{1/2}(\pi n^2+2qTn)^{-1/4}\times\exp\left(-2iTF\left(\frac{\pi n}{2qT}\right)+\frac{\pi in}{q}-\frac{\pi i}{4}-\left(G\sinh^{-1}\left(\frac{\pi n}{2qT}\right)\right)^2\right)+O((q/nT)^{1/2}),$$

where

$$F(x)=\sinh^{-1}(x^{1/2})+(x(x+1))^{1/2}.$$

These error terms contribute to the sum the amount of $\ll q^{1/2}G^{-1}l^2$, because of [4, Lemma 5].

Collecting these and using partial summation, we get