42. On the Homology Groups of the Mapping Class Groups of Orientable Surfaces with Twisted Coefficients

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(Communicated by Kunihiko KODAIRA, M. J. A., April 14, 1986)

1. Introduction. Let Σ_g be a closed orientable surface of genus g and let $\mathcal{M}_g=\pi_0 Diff_+\Sigma_g$ be its mapping class group. Also let $\mathcal{M}_{g,*}$ and $\mathcal{M}_{g,1}$ respectively be the mapping class groups of Σ_g relative to the base point $*\in\Sigma_g$ and an embedded disc $D^2\subset\Sigma_g$. It is known that these groups are perfect for all $g\geqq 3$ (see [2, 3]) and Harer determined the second homology group of them in his fundamental paper [2]. The purpose of the present note is to announce our results on the homology groups of them with coefficients in the first homology group $H_1(\Sigma_g, Z)$ of Σ_g on which the mapping class groups act naturally.

2. Low dimensional homologies. First we consider the first homology. The results of our previous paper [7] imply

Theorem 1. (i) $H_1(\mathcal{M}_g; H_1(\Sigma_g, Z))\cong Z/2g-2 \quad (g\geqq 2)$.

(ii) $H_1(\mathcal{M}_{g,1}; H_1(\Sigma_g, Z))\cong H_1(\mathcal{M}_{g,*}; H_1(\Sigma_g, Z))\cong Z \quad (g\geqq 2)$.

These groups are detected by the crossed homomorphism $f:\mathcal{M}_{g,*}\times H_1(\Sigma_g, Z)\to Z$ defined in [7]. Next the second homology group is given by the following Theorem which is one of our main results.

Theorem 2. (i) *$H_2(\mathcal{M}; H_1(\Sigma_g, Z))=0$ for all $g\geqq 12$, where $\mathcal{M}$ stands for any of $\mathcal{M}_g$, $\mathcal{M}_{g,*}$ or $\mathcal{M}_{g,1}$.*

(ii) *$H_2(\mathcal{M}; H_1(\Sigma_g, Q))=0$ for all $g\geqq 9$, where $\mathcal{M}$ is the same as above.*

Corollary 3. $H^2(\mathcal{M}_g; H^1(\Sigma_g, Z))\cong Z/2g-2 \quad (g\geqq 9)$.

The group $H^2(\mathcal{M}_g; H^1(\Sigma_g, Z))$ has the following geometric meaning. Choose a generator $o\in H^2(\mathcal{M}_g; H^1(\Sigma_g, Z))$. To any oriented differentiable Σ_g-bundle $\pi: E\to X$, we have associated in [8] a family of Jacobian manifolds $\pi': J'\to X$, which is a *flat* T^{2g}-bundle over X with structure group $H_1(\Sigma_g, Z/2g-2)\rtimes Sp(2g, Z)$, and a fibrewise embedding $j': E\to J'$ which induces an isomorphism on the first integral homology on each fibre (topological version of Earle's embedding theorem [1]). We have

Proposition 4 (compare with [1], §8). *Let $\pi: E\to X$ be an oriented Σ_g-bundle. Then the associated family of Jacobian manifolds $\pi': J'\to X$ has a cross-section if and only if $h^*(o)$ vanishes in $H^2(\pi_1(X); H^1(\Sigma_g, Z))$ where $h:\pi_1(X)\to\mathcal{M}_g$ is the holonomy homomorphism of the given Σ_g-bundle and $\pi_1(X)$ acts on $H^1(\Sigma_g, Z)$ naturally.*

Corollary 5. *The natural homomorphism $\pi:\mathcal{M}_{g,*}\to\mathcal{M}_g$ induces an isomorphism $H_3(\mathcal{M}_{g,*}, Z)\cong H_3(\mathcal{M}_g, Z)$ for all $g\geqq 10$. (It is easy to show that the homomorphism $H_3(\mathcal{M}_{g,*}, Z)\to H_3(\mathcal{M}_g, Z)$ is surjective for all $g\geqq 3$.)*