39. On a Criterion for Hypoellipticity

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(Communicated by Kôsaku YOSIDA, M. J. A., April 14, 1986)

Introduction and main theorems. In this note we give a sufficient condition for second order differential operators to be hypoelliptic. The condition is also necessary for a special class of differential operators.

Let Ω be an open set in R^n and let $P=p(x, D_x)$ be a second order differential operator with real valued coefficients in $C^\infty(\Omega)$. Let (u, v) denote the inner product of u, v in L^2 and $\|u\|^2=(u, u)$. Let $\|\cdot\|_s$ denote the Sobolev space H_s for real s.

Theorem 1. *Assume that for any* $\varepsilon>0$ *and any compact set* K *of* Ω *there is a constant* $C_{\varepsilon,K}$ *such that*

$$(1)\qquad \|(\log\langle D_x\rangle)^2 u\| \leqq \varepsilon\|Pu\| + C_{\varepsilon,K}\|u\|, \qquad u\in C_0^\infty(K),$$

where $\log\langle D_x\rangle$ *denotes a pseudodifferential operator with a symbol* $\log\langle\xi\rangle$, $\langle\xi\rangle^2=|\xi|^2+1$. *Assume that the estimate*

$$(2)\qquad \sum_{j=1}^n (\|P^{(j)}u\|^2 + \|P_{(j)}u\|_{-1}^2) \leqq C(\mathrm{Re}\,(Pu, u)+\|u\|^2), \qquad u\in C_0^\infty(K)$$

holds for a constant $C=C_K$, *where* $P^{(j)}=\partial_{\xi_j}p(x,\xi)$ *and* $P_{(j)}=D_{x_j}p(x,\xi)$. *Then* P *is hypoelliptic in* Ω. *Furthermore we have* $\mathrm{WF}\,Pu=\mathrm{WF}\,u$ *for* $u\in\mathcal{D}'(\Omega)$.

We remark that the hypothesis of (2) is removable if the principal symbol of P is non-negative. The estimate (1) is not always necessary for the hypoellipticity. We have a counter example $D_{x_1}^2+\exp(-1/|x_1|^\delta)D_{x_2}^2$ for $\delta\geqq 1$ given by [1] (cf. [6]). However, for a class of differential operators, the estimate (1) is necessary to be hypoelliptic. The result is extendible to operators of higher order. Let m be an even positive integer and let P_0 be a differential operator of the form

$$(3)\qquad P_0=D_t^m+\mathcal{A}(x, D_x) \qquad \text{in } R_t\times R_x^n,$$

where $\mathcal{A}(x, D_x)$ is a differential operator of order m with C^∞-coefficients and formally self-adjoint in an open set Ω of R_x^n. We assume that $\mathcal{A}(x, D_x)$ admits a positive self-adjoint realization $(A, D(A))$ in $L^2(\Omega)$.

Theorem 2. *Let* P_0 *be the operator defined above. Assume that* P_0 *is hypoelliptic in* $R_t\times\Omega$. *Then for any* $(t_0, x_0)\in R_t\times\Omega$ *one can find a neighborhood* ω *of* x_0 *satisfying the following: For any* $\varepsilon>0$ *there is a constant* C_ε *such that*

$$(4)\qquad \|(\log\langle D_t, D_x\rangle)^{m/2}u\|^2 \leqq \varepsilon\,\mathrm{Re}\,(P_0u, u)+C_\varepsilon\|u\|^2, \qquad u\in C_0^\infty(R_t\times\omega).$$

We remark that when $m=2$ the estimate (1) follows from (4) by means of the partition of unity over K and the replacement of u by $(\log\langle D_t, D_x\rangle)u$.