31. On the Crepant Blowing-Ups of Canonical Singularities and Its Application to Degenerations of Surfaces

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Let X be a normal algebraic variety over $\boldsymbol{C}$, and let D be a Weil divisor on it. We would like to know when the sheaf of graded $\mathcal{O}_X$-algebras $\mathcal{R}(D) := \oplus_{m\geqq 0} \mathcal{O}_X(mD)$ is finitely generated, where the $\mathcal{O}_X(mD)$ are reflexive sheaves of rank 1 corresponding to the mD. It is equivalent to saying that there exists a projective morphism $f: X'\to X$ which is an isomorphism in codimension 1 and such that the strict transform D' of D on X' is $\boldsymbol{Q}$-Cartier and f-ample. The problem is trivial in case $\dim X=2$; f must be an isomorphism and the condition for the finite generatedness is simply that D is $\boldsymbol{Q}$-Cartier. It is well known that a normal surface singularity X is (analytically) $\boldsymbol{Q}$-*factorial*, i.e., an arbitrary (analytic) Weil divisor on X is $\boldsymbol{Q}$-Cartier, if and only if X is a rational singularity. In this paper we announce a partial generalization of this fact to 3-dimensional case. (We refer the reader to [3] for definitions concerning minimal models.)

Theorem 1. *Let X be a 3-dimensional normal algebraic variety over $\boldsymbol{C}$ which has at most canonical singularities, and let D be a Weil divisor on it. Then $\mathcal{R}(D)$ is finitely generated.*

We note that a rational Gorenstein singularity is canonical. The theorem is proved in the following way. Let X be as in Theorem 1 and let $\mu: Y\to X$ be a desingularization. Then we can write $K_Y=\mu^*K_X+\sum_j a_jF_j$ with $a_j\geqq 0$ by definition, where the F_j are exceptional divisors of μ. We define $e(X)$ as the number of divisors F_j for which μ is *crepant*, i.e., $a_j=0$ (it is easy to see that $e(X)$ does not depend on the choice of μ). For example, $e(X)=0$ if and only if X has at most terminal singularities. We define also $\sigma(X) := \dim_{\boldsymbol{Q}} Z_2(X)_{\boldsymbol{Q}}/\mathrm{Div}\,(X)_{\boldsymbol{Q}}$, where $Z_2(X)_{\boldsymbol{Q}}$ and $\mathrm{Div}\,(X)_{\boldsymbol{Q}}$ are groups of $\boldsymbol{Q}$-divisors and $\boldsymbol{Q}$-Cartier divisors, respectively (one can prove that $\sigma(X)$ is finite). Thus X is $\boldsymbol{Q}$-factorial if and only if $\sigma(X)=0$. Our theorem is proved by induction on $e(X)$ and $\sigma(X)$ in the category consisting of varieties X' with projective birational morphisms $f: X'\to X$ which are *crepant*, i.e., $K_{X'}=f^*K_X$; e.g., an isomorphism in codimension 1 is crepant, since K_X is $\boldsymbol{Q}$-Cartier. Theorem 1 in case $e(X)=0$ is proved by using Brieskorn's flips as in [5]. The termination of log-flips in case $e(X')=n$ produces the existence of the log-flip in case $e(X')=\mathrm{n}+1$ (cf. [3]). In the course of the proof, the concept of the *sectional decomposition*, which is a rather trivial generalization of the Zariski decomposition for surfaces (cf.