30. A Topology on Arithmetical Lattice-Ordered Groups†)

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The aim of this note is to describe explicitly the weakest topology on an arithmetical lattice-ordered group for which given lattice-ideal is open. This topology is utilized to treat some of ring topologies, field topologies and $\mathfrak{p}$-adic topologies.

1. A lattice-ordered group (abbr. *l.o.* group) $G=(G, \cdot, \leq)$ is called *arithmetical*, if it is a conditionally complete lattice and a free group generated by the set P of all prime elements in the cone (integral part) of G. Then G is abelian by Iwasawa's theorem for *l.o.* groups [1], [2], so that each element a of G has a unique factorization in the form:

$$a=\prod_{p\in P} p^{\nu(p,a)}, \qquad \nu(p, a)\in \boldsymbol{Z}$$

where $\boldsymbol{Z}$ is the integers and $\nu(p, a)$ is the exponent of a at p. This factorization was generalized by the author [4] as follows. Each lattice-ideal (abbr. *l*-ideal) J of G has a unique factorization in the form:

$$J=\prod_{p\in P_+(J)} J(p)^{\nu(p,J)}\cdot \bigcup \prod_{p\in P_-(J)} J(p)^{\nu(p,J)}\cdot e_{P_{-\infty}(J)}$$

where $\prod$ is finite product, $\cup$ is set-theoretical union, $J(p)$ is the principal *l*-ideal generated by p, $\nu(p, J)=\inf\{\nu(p, a)\,;\, a\in J\}$, $P_+(J)=\{p\in P\,;\, 0<\nu(p, J)\}$, $P_-(J)=\{p\in P\,;\, -\infty<\nu(p, J)<0\}$, $P_{-\infty}(J)=\{p\in P\,;\, \nu(p, J)=-\infty\}$, $e_{P_{-\infty}(J)}$ is the $P_{-\infty}(J)$-component of unit e of G. (If J is principal, $P_-(J)$ is finite and $e_{P_{-\infty}(J)}$ coincides with the cone of G.)

A non-void set $\boldsymbol{U}$ of *l*-ideals of G is called a *u-system* of G if it satisfies the following conditions:

1) If $J_1, J_2\in \boldsymbol{U}$, there is $J_3\in \boldsymbol{U}$ such that $J_3\subseteq J_1\cap J_2$.
2) If $a\in G$, $J_1\in \boldsymbol{U}$, there is $J_2\in \boldsymbol{U}$ such that $aJ_2\subseteq J_1$.
3) If $J_1\in \boldsymbol{U}$, there is $J_2\in \boldsymbol{U}$ such that $J_2J_2\subseteq J_1$.

Then $\boldsymbol{U}$ determines a topology on G, which is called an *l-ideal topology* on G. In symbol: $\mathrm{T}(\boldsymbol{U})$. Let $g(n\,;p, J)$ be the integer m or $-\infty$ such that $\nu(p, J)/2^n\leq m<\nu(p, J)/2^n+1$. We define

$$J^{(n)}=\bigcup\prod_{p\in P_-(J)} J(p)^{g(n;p,J)}\cdot e_{P_{-\infty}(J)}$$

for $n\in \boldsymbol{N}_0$, the non-negative integers. Then since (1°) $J^{(n)}\supseteq J^{(n+1)}$, (2°) $J^{(n)}\supseteq e_{P_{-\infty}(J)}$, (3°) $J^{(n)}\supseteq J^{(n+1)}J^{(n+1)}$ and (4°) $(J^{(n)})^{(m)}=J^{(n+m)}$, we can show that

$$\boldsymbol{U}(J)=\{aJ^{(n)}\,;\, a\in G,\ n\in \boldsymbol{N}_0\}$$

forms a u-system of G.

Theorem 1. *Let J be an l-ideal of G. Then among the set of all*

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†) Dedicated to Emeritus Professor Hidetaka TERASAKA for his octogenarian birthday.