23. *The Grothendieck Conjecture and Padé Approximations**)

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§0. The Grothendieck conjecture [1], [2] predicts the global algebraic behavior of solutions of linear differential equations, provided that these equations have "sufficiently many solutions" after reduction (mod p) for almost all p. In-depth studies of this conjecture and its interesting generalizations belong to Katz [1], [3]. However, the conjecture remains open in many important cases. One of the crucial cases, pointed out in [1], [4], is the case of Lamé-type equations or the case of rank one equations over an elliptic curve. In this case, we show how the methods of Padé approximations can be used to prove the Grothendieck conjecture in this and other important cases.

§1. For expositions of the p-adic properties of linear differential equations connected with the Grothendieck conjecture see [1], [2], [3], [4]. If a linear differential equation is represented in a matrix form

$$(d/dx)\vec{f}+A(x)\vec{f}=0, \tag{1}$$

with $A\overset{\text{def}}{=}A(x)\in M(n,K(x))$ and an algebraic number field K, then the p-curvature operator Ψ_p of (1) mod p is $\Psi_p=((d/dx)\cdot I+A)^p(\text{mod}\ p)$.

Here Ψ_p is, in fact, a linear operator: $\Psi_p=A_p$ (mod p), where $A_1=A$, $A_{n+1}=(d/dx)A_n+AA_n$.

The Grothendieck conjecture. For a system (1), $\Psi_p=0$ for almost all p if and only if all solutions of (1) are algebraic functions. Detailed studies of equivalents of the Grothendieck conjecture are presented in Honda [2] for scalar linear differential equations

$$Lf\overset{\text{def}}{=}a_nf^{(n)}+\cdots+a_1f'+a_0f=0 \tag{2}$$

and $a_i=a_i(x)\in K[x]$ $(0\le i\le n)$. Let, for a prime ideal ρ of K, $\overline{K}_\rho$ denotes the residue field and L_ρ denotes the reduction mod ρ of L. Other reformulations of the Grothendieck conjecture are the following:

1) If for almost all prime ideals ρ, $L_\rho f=0$ has n solutions in $\overline{K}_\rho(x)$ which are independent over $\overline{K}_\rho(x^p)$, then all solutions of (2) are algebraic functions;

2) If for almost all p we have $(d/dx)^p\equiv 0 \bmod K_\rho((x))[d/dx]L_\rho$, then all solutions of (2) are algebraic functions.

A condition weaker than assumptions of the Grothendieck conjecture is the condition of global nilpotence of (1), i.e., the condition of nilpotence of matrices Ψ_p for almost all p [1], [2].

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*) To Professor Bers on his 70th Birthday.