13. A Formula for the Number of Semi-simple Conjugacy Classes in the Arithmetic Subgroups

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0. The purpose of this note is to present a general formula for the number of conjugacy classes in the arithmetic subgroups of a reductive algebraic group G defined over an algebraic number field k, of those elements which are contained in a single semi-simple conjugacy class of G_k.

It is known that the semi-simple conjugacy classes in the classical groups are parametrized by isomorphism classes of various kinds of hermitian forms. Moreover, the centralizer of the elements in a class is the unitary group of the corresponding hermitian forms (c.f. [1], [5], [8], [9], [10]). From this fact, one can deduce the *Hasse-Principle* for the conjugacy classes in the classical groups defined over algebraic number fields ([1], [5]). Now it seems natural to expect that, also for the groups over the ring of integers, the sets of conjugacy classes are parametrized by isometric classes of hermitian forms, so that their numbers are counted by the class numbers of such forms. In fact our previous paper [6] proves that this is exactly the case for a class of involutive elements in the Siegel modular group $\mathrm{Sp}(n, Z)$. The formula we shall give here shows that this is also true in the most general sense: it expresses the number of our conjugacy classes as a sum of the class numbers of the centralizer, up to the local factors which turn out to be one in most cases.

Notation. We write $g_1 \underset{H}{\sim} g_2$ if g_1, g_2 are H-conjugate for a subgroup H of G, and denote by $G/\!/H$ the set of H-conjugacy classes in G.

1. Let G be a reductive algebraic group defined over an algebraic number field k, and suppose that $G_k \subseteq GL_n(k)$. By an idèlic arithmetic subgroup U, we mean an open subgroup of G_A, the idèle group of G, which is of the form

$$U=\prod_{\mathfrak{p}} U_{\mathfrak{p}} \times G_\infty, \qquad U_{\mathfrak{p}}=G_{k_{\mathfrak{p}}} \cap GL_n(O_{\mathfrak{p}}),$$

where $G_{k_{\mathfrak{p}}}$ is the $\mathfrak{p}$-adic completion of G_k, $O_{\mathfrak{p}}$ is the ring of integers of $k_{\mathfrak{p}}$, and G_∞ is the archimedian part of G_A. By the reduction theory, it is known that G_A is decomposed as a disjoint union of finite double cosets Ug_iG_k $(1 \leq i \leq H)$, where $H=H(U)$ is the class number of U in G_A. Then the groups $\Gamma_i := G_k \cap g_i^{-1}Ug_i$ are called (global) *arithmetic subgroups* corresponding to U.

2. Let $g \in G_k$ be a semi-simple element, and put

$$\begin{aligned} &C_k(g) := \{x^{-1}gx\,;\, x \in G_k\} && (G_k\text{-conjugacy class of } g) \\ (1) \quad &Z_G(g) := \{x \in G_k\,;\, xg=gx\} && (\text{centralizer of } g \text{ in } G_k) \end{aligned}$$