10. *Commutators on Dyadic Martingales*

By J.-A. CHAO*) and H. OMBE**)

(Communicated by Kôsaku YOSIDA, M. J. A., Feb. 12, 1985)

§1. **Introduction.** A characterization of $BMO(\boldsymbol{R}^n)$ by commutators with singular integrals was given by Coifman-Rochberg-Weiss [7]. (See also [8].) Later, an analogue for regular martingales is shown by Janson [9]. Recently, Chanillo [3] and Rochberg-Weiss [11] and Komori [10] obtained a similar result on commutators with fractional integrals. It is the purpose of this note to study fractional integrals and commutators in the dyadic martingale setting. A version of fractional integrals I^α for dyadic martingales is introduced which is parallel to that on Walsh-Fourier series studied by Watari [14], and that on local fields by Taibleson [13]. The boundedness of commutators $[b, I^\alpha]$ shall be used to characterize the multiplicating function b.

§2. **Fractional integrals.** Let $\mathcal{F}_n$ be the sub-σ- field generated by dyadic intervals of length 2^{-n} in $[0,1]$, $n=0,1,2,\cdots$. A martingale $\{f_n\}_{n\geq 0}$ relative to $\{\mathcal{F}_n\}_{n\geq 0}$ is a dyadic martingale. For an integrable function f on $[0,1)$, the conditional expectations $f_n\equiv E(f\,|\,\mathcal{F}_n)$, $n=0,1,2,\cdots$, form a dyadic martingale whose L^p norm, $\sup_n \|f_n\|_p$, equals to the L^p norm of the function f, for $p\geq 1$. We shall identify f with $\{f_n\}$ by writing $f=\{f_n\}$ and assume $f_0=0$. Let $\{d_n\}$ be the difference sequence of $f=\{f_n\}$, i.e. $f_n=\sum_{k=1}^n d_k$. The maximal function and square function of $f=\{f_n\}$ are given by $f^*=\sup|f_n|$ and $S(f)=(\sum_{k=1}^\infty d_k^2)^{1/2}$, respectively. The following are well-known. (See [1], [2] and [5].)

$$(1)\qquad \begin{aligned} &\|f^*\|_p\approx\|f\|_p\approx\|S(f)\|_p, && \text{for } 1<p<\infty, \text{ and}\\ &\|f^*\|_p\approx\|S(f)\|_p, && \text{for } 0<p<\infty.\end{aligned}$$

Now for a dyadic martingale $f=\{f_n\}$ and $\alpha\in\boldsymbol{R}$, we define the fractional integral $I^\alpha f=\{(I^\alpha f)_n\}$ of f (of order α) by $(I^\alpha f)_n=\sum_{k=1}^n 2^{-k\alpha}d_k$, whose maximal function is $(I^\alpha f)^*=\sup_n|\sum_{k=1}^n 2^{-k\alpha}d_k|$. If $\alpha>0$, $I^\alpha f$ is simply a martingale transform introduced by Burkholder [1]. It is trivial that $\|(I^\alpha f)^*\|_p \leq C\|I^\alpha f\|_p\leq C\|f\|_p$ for $0<\alpha<\infty$ and $1<p<\infty$. Moreover, we have

Theorem 1. *For integrable f,*

$$(2)\qquad \|(I^\alpha f)^*\|_q\leq C\|f\|_p \qquad \textit{where } 1<p<q<\infty$$

and $\alpha=1/p-1/q$;

---

*) Department of Mathematics, Cleveland State University, Cleveland, OH 44115.
Partly supported by grants from the National Science Foundation and Cleveland State University. 1980 mathematics subject classification; 60G46; 42A45; 60G42.

**) Department of Mathematical Sciences, University of Texas at El Paso, El Paso, TX 79968.