9. *Propagation of Singularities of Solutions to Semilinear Schrödinger Equations*

By Tsutomu SAKURAI
Department of Pure and Applied Sciences, University of Tokyo

(Communicated by Kôsaku YOSIDA, M. J. A., Feb. 12, 1985)

The purpose of this note is to study micro-local singularities of solutions to some semilinear Schrödinger equation. In [4], Rauch studied singularities of classical solutions to the equation $\Box u=f(u)$ and showed that singularities modulo H^r, for some r, propagate along null bicharacteristic strips. Here, we follow his arguments and obtain a similar result for semilinear Schrödinger equations.

1. Notation and statement of the result. Let Ω denote an open set of $\boldsymbol{R}^n$. Let $M=(\mu_1, \cdots, \mu_n)$ be a multiweight on the dual space $\boldsymbol{R}_n$, with $\inf\{\mu_j\}=1$. If $\xi \in \boldsymbol{R}_n$ and $t>0$ we shall use the notation $t^M\xi=(t^{\mu_1}\xi_1, \cdots, t^{\mu_n}\xi_n)$. We shall say that a function g on $\Omega\times(\boldsymbol{R}_n\backslash 0)$ is (M-) quasi-homogeneous of degree m if $g(x, t^M\xi)=t^m g(x, \xi)$ for $t>0$, and that a subset Γ of $\Omega\times(\boldsymbol{R}_n\backslash 0)$ is a M-cone if $(x, \xi)\in\Gamma$ implies $(x, t^M\xi)\in\Gamma$ for every $t>0$. We introduce the function $[\cdot]_M$ defined implicitly by $\sum \xi_j^2/[\xi]_M^{2\mu_j}=1$ if $\xi\neq 0$ and $[0]_M=0$.

We let $S^m_M(\Omega)$ denote the space of C^∞-functions $p: \Omega\times\boldsymbol{R}_n\to\boldsymbol{C}$ satisfying the following estimate: for every $\alpha, \beta \in \boldsymbol{N}^n$, $K\subset\subset\Omega$ there exists a constant $C=C_{\alpha\beta K}$ such that

$$|\partial_\xi^\alpha\partial_x^\beta p(x, \xi)|\leq C(1+[\xi]_M)^{m-\langle\alpha, M\rangle} \qquad \text{for } x\in K,$$

where $\langle\alpha, M\rangle=\sum\alpha_j\mu_j$. If $p\in S^m_M(\Omega)$ we set

$$p(x, D_x)u(x)=(2\pi)^{-n}\iint e^{i\langle x-y, \xi\rangle}p(x, \xi)u(y)dyd\xi \qquad \text{for } u\in C_0^\infty(\Omega),$$

and use the terminology of *M-pseudo-differential operators* for it. We shall say that $p\in S^m_M(\Omega)$ is a classical symbol if p has an asymptotic expansion by quasi-homogeneous functions p_{m_j} of degree m_j: $p(x, \xi)\sim p_m(x, \xi)+\sum_{j=1}^\infty p_{m_j}(x, \xi)$, with $m-1\geq m_1>m_2>\cdots$. For a classical symbol $p\in S^m_M(\Omega)$ we call the top term p_m principal symbol and define its M-Hamiltonian vector field in $\Omega\times(\boldsymbol{R}_n\backslash 0)$ to be $\sum_{\mu_j=1}(\partial_{\xi_j}p_m\partial_{x_j}-\partial_{x_j}p_m\partial_{\xi_j})$ which is denoted by H_p^M. To the classical M-pseudo-differential operator with real principal symbol, a bicharacteristic strip is an integral curve of the M-Hamiltonian vector field.

Let $H^s_M(\Omega)$ be a weighted Sobolev space with the norm

$$\|u\|_{M,s}=\|(1+[\xi]_M)^s\hat{u}(\xi)\|_{L^2} \qquad \text{for } u\in C_0^\infty(\Omega).$$

We also define its micro-localization:

Definition. Let $u\in\mathcal{D}'(\Omega)$ and $z_0\in\Omega\times(\boldsymbol{R}_n\backslash 0)$. The implication $u\in H^s_M(z_0)$ means that there exists a classical symbol $a(x, \xi)\in S^0_M(\Omega)$ such that $a_0(z_0)\neq 0$ and $a(x, D_x)u\in H^s_M(\Omega)$. (We then say that u belongs to H^s_M at z_0.)