83. Some Results on Bessel Processes

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Let α be any real number. The Bessel process with index α is the diffusion process on the half line $\boldsymbol{R}^+=[0, \infty)$, whose infinitesimal generator agrees with the differential operator

$$A=\frac{1}{2}\left(\frac{d^2}{dx^2}+\frac{\alpha-1}{x}\cdot\frac{d}{dx}\right). \tag{1}$$

We note that the formula (1) for the generator implies that the boundary point 0 is

entrance not exit for $\alpha\geqq 2$
entrance and exit for $0<\alpha<2$
exit not entrance for $\alpha\leqq 0$.

Thus for $\alpha\geqq 2$ or $\alpha\leqq 0$, the processes are completely specified by the generator (1) above, but for $0<\alpha<2$ appropriate boundary condition must be imposed at the origin. In this note we deal with two types of the boundaries, i.e., reflecting barrier and absorbing barrier.

Following Yosida [4], we define the generalized potential operator V for the semigroup T_t by

$$Vf=\lim_{\lambda\downarrow 0}\int_0^\infty e^{-\lambda t}T_t f\,dt \qquad (\lambda>0). \tag{2}$$

The representation of the potential operators associated with Bessel processes which is shown in [1], [2], will be stated here as

Proposition 1 (Reflecting case: Theorem 3 of Arakawa-Takeuchi [1]). *Assume that $xf(x)\in L^1(\boldsymbol{R}^+)$ for $\alpha>0$, $\alpha\neq 2$ and $xf(x)\log x\in L^1(\boldsymbol{R}^+)$ for $\alpha=2$.*

(i) *For $0<\alpha\leqq 2$, a necessary and sufficient condition for $f\in\mathcal{D}(V)$ is*

$$\int_0^\infty x^{\alpha-1}f(x)dx=0. \tag{3}$$

If $f\in\mathcal{D}(V)$, then we have

$$Vf(x)=2\int_0^\infty U(x\vee y)y^{\alpha-1}f(y)dy \tag{4}$$

with the kernel

$$U(x)=\begin{cases}\dfrac{1}{\alpha-2}\cdot\dfrac{1}{x^{\alpha-2}} & \text{if } 0<\alpha<2 \text{ and } \alpha>2\\[2mm] \log\dfrac{1}{x} & \text{if } \alpha=2,\end{cases} \tag{5}$$

here $x\vee y$ denotes the greater of x and y.

(ii) *For $\alpha>2$, a function f such that $xf(x)\in L^1(\boldsymbol{R}^+)$ is contained in $\mathcal{D}(V)$, and $Vf(x)$ is expressed by* (4) *with* (5).

Proposition 2 (Absorbing case: Takeuchi [2]). *For $-\infty<\alpha<2$, a*