79. *Dedekind Domains which are not obtainable as Finite Integral Extensions of PID*

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All known Dedekind domains are obtainable as the integral closure of a suitable PID R in a finite extension of the quotient field of R. But the converse is not the case, as we shall see in this note (cf. Zariski-Samuel, [2], Chap. V, § 8).

§ 1. An example. Let K be a quadratic extension of the field of rational numbers $\boldsymbol{Q}$ whose class number is greater than 1. Let p be a rational prime number which is the product of two distinct prime elements in K, say, π and π' : $p=\pi\pi'$. The density theorem of prime ideals assures the existence of such a prime number.

Let S be the set of the elements π^n $(n\geqq 1)$. The set S is a multiplicative set of the ring A of algebraic integers in K. Let A_S be the quotient ring of A with respect to the set S. Then the ring $A_S=R$ is a Dedekind domain, since A is a Dedekind domain. It is easily seen that $A_S\cap \boldsymbol{Q}=\boldsymbol{Z}$ and the integral closure of $\boldsymbol{Z}$ in K is the ring A, which is a proper subring of A_S.

Next we show that the ring A_S is not a PID. For this purpose it suffices to prove that the ideal class group of A_S is isomorphic with that of A, which is the ideal class group of the field K. Let I_0 be the semigroup of ideals of A prime to the ideal $A\pi$, and I_S the semigroup of ideals of A_S. Consider the mapping $\mathfrak{a}\to\mathfrak{a}A_S$. This is clearly a bijection of I_0 onto I_S. Suppose $\mathfrak{a}A_s=(\alpha/\pi^k)A_S$ for some $\alpha\in A$ and a positive integer k. Since $\mathfrak{a}\subseteq(\alpha/\pi^k)A_S$, we have $\pi^s\mathfrak{a}\subseteq\alpha A$ for some integer s. As A is a Dedekind domain, there exists an ideal $\mathfrak{b}$ of A such that

$$\pi^S\mathfrak{a}=\alpha\mathfrak{b}. \tag{1}$$

Since $\alpha/\pi^k\in\mathfrak{a}A_S$, we have $\pi^t\alpha A\subseteq\mathfrak{a}$ for some integer t. By the same reason as above, we have an ideal $\mathfrak{c}$ of A satisfying

$$\pi^t\alpha A=\mathfrak{a}\mathfrak{c}. \tag{2}$$

From (1) and (2) we obtain $\pi^m A=\mathfrak{b}\mathfrak{c}$ for some m. This implies that the ideal $\mathfrak{b}$ divides the principal ideal $\pi^m A$. Thus we see that $\mathfrak{b}$ is principal, and so is $\mathfrak{a}$. Since any ideal class of A has a representative which is prime to $A\pi$, we have proved that the ideal class group of K is isomorphic with the ideal class group of A_S. Thus we have the following:

There exists a Dedekind domain R which is not obtained as the integral closure of $R\cap F$ in the quotient field K of R for any proper subfield F of K.