29. A Generalization of Gauss' Theorem on the Genera of Quadratic Forms*)

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Let T be a torus defined over $\boldsymbol{Q}$. As is well known, one can associate with T the class number h_T independently of matrix representation of T. (See [2] p. 119 footnote and p. 120 line 17. As for basic facts on tori, see [2], [3].) When $T=R_{K/\boldsymbol{Q}}(G_m)$, the multiplicative group $K^\times$ of an algebraic number field K viewed as an algebraic group over $\boldsymbol{Q}$, h_T coincides with the ordinary class number h_K of the field K. Consider a short exact sequence of tori over $\boldsymbol{Q}$:

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0.$$

It is natural to think of the alternating product

$$\frac{h_T}{h_{T'} h_{T''}}.$$

In his thesis Shyr considered this problem, obtained a general formula using [2], [3] and noticed, among others, that the formula is nothing but the formula of Gauss

(G) $$h_K^+ = h_K^* 2^{t-1}$$

when applied to $T=R_{K/\boldsymbol{Q}}(G_m)$, $T''=G_m$ and $T'=$the kernel of the norm map $N: T \to T''$, where $K/\boldsymbol{Q}=$a quadratic extension, $h_K^+=$the class number of K in the narrow sense, $h_K^*=$the number of classes in a genus and $t=$the number of rational primes ramified in K/Q. (See [4] and [5].)

In this note, we shall report formulas of the same type as (G) for any cyclic Kummer extension K/k and clarify the relationship between ingredients of our formula and those appearing in the classical treatment of class field theory.

So, let k be an algebraic number field of degree n_0 over $\boldsymbol{Q}$ which contains a primitive n-th root of 1 ($n \geqq 2$) and K/k be a cyclic extension of degree n. Consider tori $T_0=R_{K/k}(G_m)$, $T_0''=G_m$ over k and the exact sequence over k:

$$0 \longrightarrow T_0' \longrightarrow T_0 \longrightarrow T_0'' \longrightarrow 0$$

where T_0' is the kernel of the norm map $N: T_0 \to T_0''$. Applying $R_{k/\boldsymbol{Q}}$, we obtain the exact sequence over $\boldsymbol{Q}$:

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0$$

where $T=R_{k/\boldsymbol{Q}}(T_0)=R_{K/\boldsymbol{Q}}(G_m)$, $T''=R_{k/\boldsymbol{Q}}(G_m)$ and $T'=R_{k/\boldsymbol{Q}}(T_0')$. We have $h_T=h_K$, $h_{T''}=h_k$. As for the Tamagawa numbers, we have $\tau(T)=\tau(T'')=1$ and $\tau(T')=\tau_k(T_0')=n$ since K/k is cyclic of degree n. (See [3] Corollary to

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