23. *Transmutation, Filtering, and Scattering*

By Robert CARROLL
Department of Mathematics, University of Illinois

(Communicated by Kôsaku YOSIDA, M. J. A., March 12, 1984)

Abstract. It is shown that in suitable circumstances the characterization of transmutation kernels via minimization can be achieved via stochastic information and accomplishes the same thing in stochastic geometry as linear least squares estimation.

1. Introduction. We will show here how the two areas of transmutation and linear filtering theory are directly connected by a minimization principle. Thus we refer first to transmutation theory as developed in [1], [2] for example where the basic theme is to study operators $B: P \to Q$ intertwining P and Q ($BP=QB$ acting on suitable functions); P and Q are two second order ordinary differential operators and B is generally an integral operator with a distribution kernel. Such transmutations B are often characterized by their action on suitable eigenfunctions φ_λ^P of P ($P\varphi_\lambda^P = -\lambda^2 \varphi_\lambda^P$) and $\varphi_\lambda^Q = B\varphi_\lambda^P$ satisfies (*) $Q\varphi_\lambda^Q = -\lambda^2\varphi_\lambda^Q$; they play an important role in the study of special functions, eigenfunction integral transforms, inverse problems, etc. In particular in classical quantum scattering theory with $P=D^2$ and $Q=D^2-q(x)$ extensive use of transmutation methods appears in the physics literature (cf. [8]). We take this as our basic situation here also, in establishing links with estimation theory, and take $\varphi_\lambda^P(x)=\mathrm{Cos}\,\lambda x$ with $\varphi_\lambda^Q(x)$ defined to satisfy (*) with $\varphi_\lambda^Q(0)=1$ and $D_x\varphi_\lambda^Q(0)=h \neq 0$. Then

$$\varphi_\lambda^Q(y) = (B\varphi_\lambda^P)(y) = \mathrm{Cos}\,\lambda y + \int_0^y K(y,x)\,\mathrm{Cos}\,\lambda x\,dx$$

and $K(y,x)$ is a function with smoothness depending on q. Strictly speaking one should index with h, i.e. B_h, K_h, etc. but we omit the index h for simplicity. Also assume the spectral theory for Q ($=Q_h$) is based on a measure $d\omega(\lambda)=\omega d\lambda$ (no bound states). Now recently in [3]–[6] it was shown that various transmutations can be characterized by minimization with Gelfand-Levitan (G-L) or Marčenko (M) equations arising as Euler equations (cf. also [10]). In the same spirit K ($=K_h$) above will arise from minimizing

$$\Xi = \int_0^T \int_0^\infty \left\{ \varphi_\lambda^Q(y) - \mathrm{Cos}\,\lambda y - \int_0^y \mathfrak{K}(y,x)\,\mathrm{Cos}\,\lambda x\,dx \right\}^2 d\omega dy \tag{1.1}$$

($T<\infty$ fixed) over a suitable class of kernels $\mathfrak{K}$ having the same properties as K above. For questions of linear estimation, prediction,